Group Incentive Compatibility in a Market with Indivisible Goods: A Comment

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Abstract

We note that the proofs of Bird (1984), the first to show group strategy-proofness of top trading cycles (TTC), require correction. We provide a counterexample to a critical claim and present corrected proofs in the spirit of the originals. We also present a novel proof of strong group strategy-proofness using the corrected results.

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1 Introduction

Strategy-proof mechanisms are desirable because they are immune to an individual agent's misrepresentations. Agents' decisions are thus straightforward, because the optimal action for any individual is to report his or her true preferences. Group strategy-proof mechanisms ensure this is also true for coalitions of agents, protecting less sophisticated and less well-connected agents.

Therefore, in problems such as school assignment, housing assignment, and organ exchange, group strategy-proofness is valuable. These settings are applications of the canonical house swapping model, where each agent is endowed with a single indivisible good and has strict preferences over the set of goods. In this model, the top trading cycles (TTC) mechanism of Shapley and Scarf (1974) produces the strong core allocation. Roth and Postlewaite (1977) show that this is also the unique competitive equilibrium allocation.

Roth (1982) shows that TTC is strategy-proof. Bird (1984) presents a proof that TTC is weakly group strategy-proof. In this note, we show that Bird's proof requires correction. To our knowledge, we are the first to do so. While others have since provided alternative proofs that TTC is group strategy-proof,[1] we present new proofs in the spirit of the originals in Bird (1984). We also prove a non-obvious claim about strong group strategy-proofness, which Bird (1984) presents as a corollary.

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[1] See Moulin (1995) Lemma 3.3 and Pápai (2000).

In the next section, we present the model notation. In Section 3, we correct a critical claim, Lemma 1, which Bird (1984) uses to prove weak group strategy-proofness. We then revise his proof of weak group strategy-proofness. Finally, we present a new proof of strong group strategy-proofness using the corrected Lemma 1.

2 Model and notation

We retain the notation in Bird (1984) and recount it briefly here. Let $N = \{1, ..., n\}$ be the set of agents and let $w = (w_1, ..., w_n)$ be the endowment, where agent i is endowed with w_i, which we call a house. Each agent i has strict preferences P_i over the houses. We denote the weak preferences R_i. Let $P = (P_1, ..., P_n)$ be the preference profile of all agents. An allocation is a vector $x = (x_1, ..., x_n)$ where each x_i corresponds to some w_j.

Let $T(N, P)$ denote the allocation resulting from TTC applied to (N, P). For convenience, we use $TTC(N, P)$ to denote the procedure of TTC applied to (N, P). Let $S_k(P) \subseteq N$ be the set of agents in the kth trading cycle of $TTC(N, P)$, and let $S_0 = \emptyset$.[2] Define $R_k(P) = \cup_{i=1}^{k} S_i(P)$.

Suppose a subset Q of agents report their preferences as $P'_Q \neq P_Q$. Let $P' = (P'_Q, P_{-Q})$. Denote $x = T(N, P)$ and $x' = T(N, P')$.

We seek to show that TTC is weakly group strategy-proof: for any Q and P'_Q, there is some agent $i \in Q$ such that $x_i P_i x'_i$ or $x_i = x'_i$. That is, at least one agent in Q is weakly worse off under the misrepresentation. Additionally, we show TTC is strongly group strategy-proof: for any Q and P'_Q, there is some agent $i \in Q$ such that $x_i P_i x'_i$. That is, at least one agent in Q is strictly worse off under the misrepresentation.

3 Weak group strategy-proofness

Bird (1984) makes the following claim, which is critical to the main result.

Claim 1 (Bird, 1984. Lemma 1). If there is an $i \in S_k(P)$ such that $x'_i P_i x_i$, then there exist $j \in R_{k-1}(P)$ and $h \in N \setminus R_{k-1}(P)$ such that $w_h P'_j x_j$.

He gives the following intuition:

> [I]f any trader wants to get a more preferred good, he needs to get a trader in an earlier cycle to change his preference to a good that went in a later trading cycle. From this result, the group incentive compatibility follows easily.

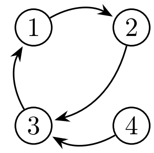
The lemma as stated requires correction. We first give a counterexample.

Example 1 (Counterexample to Claim 1). *Let $N = \{1, 2, 3, 4\}$ with the following preferences.*

[2] The order of cycles generated by TTC is not generally unique. It is possible that two or more cycles are formed at the same step. However, the results carry through under any ordering of these cycles.

P_1	P_2	P_3	P_4
w_2	w_3	w_1	w_3
w_1	w_2	w_4	w_4
		w_3	

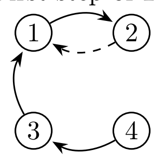
Figure 1: First step of $TTC(N, P)$



The TTC allocation is $x = (w_2, w_3, w_1, w_4)$. Now consider an alternative preference profile P':

P_1	P'_2	P_3	P_4
w_2	w_1	w_1	w_3
w_1	w_3	w_4	w_4
	w_2	w_3	

Figure 2: First step of $TTC(N, P')$



The new TTC allocation is $x' = (w_2, w_1, w_4, w_3)$.

In the notation of Claim 1, we have $i = 4$ and $k = 2$. That is, $4 \in S_2(P)$ and $x'_4 P_4 x_4$. Yet, there do not exist $j \in R_{k-1}(P) = S_1(P)$ and $h \in N \setminus R_{k-1}(P) = S_2(P)$ such that $w_h P'_j x_j$. The only candidate for j is $2 \in R_1(P) = S_1(P)$, since she is the only agent whose preferences change under P'. But she does not rank any houses from $N \setminus R_{k-1}(P) = S_2(P)$ above $x_2 = w_3$.

The error in Bird's proof of Lemma 1 stems from the following erroneous claim.

Claim 2. Let $x_m P'_m w_n$ for all $m \in R_{k-1}(P)$ and $n \in N \setminus R_{k-1}(P)$. That is, all members of the first $k-1$ cycles rank their original assignments above all houses from cycles k and later. Then $R_{k-1}(P') = R_{k-1}(P)$. That is, the set of agents assigned in the first $k-1$ cycles of $TTC(N, P')$ is the same as the set of agents assigned in the first $k-1$ cycles of $TTC(N, P)$.

The above counterexample also serves as a counterexample to Claim 2, since $R_1(P) \neq R_1(P')$.

It is not necessary for an agent in an earlier cycle $\kappa < k$ to change her preference to a house in cycle k or later. She may change her preference to a house in cycle κ or later. This is the necessary addition; we present a corrected version.

Lemma 1 (Claim 1, Corrected)**.** *If there is an $i \in S_k(P)$ such that $x'_i P_i x_i$, then there exist $j \in S_\kappa(P)$ where $\kappa < k$ and $h \in N \setminus R_{\kappa-1}(P)$ such that $w_h P'_j x_j$.*

That is, if an agent wants to get a more preferred good, he needs an agent in an earlier cycle to misrepresent her preferences to favor a good that went in her own cycle or a later cycle. Using the notation of Lemma 1, h may be in the same cycle κ as j. In the case of Example 1, $i = 4, k = 2, \kappa = 1, j = 2$, and $h = 1$.

Proof. Suppose there exist k and $i \in S_k(P)$ such that $x'_i P_i x_i$. Toward a contradiction, suppose that for each $\kappa < k$, for all $j \in S_\kappa(P)$, we have that $x_j P'_j w_h$ for all $h \in N \setminus R_{\kappa-1}(P)$ and $h \neq j$. That is, all agents in cycles before k still rank their original allocation over any other house in their own cycle or later. We show by strong induction on the cycles t of $TTC(N, P)$ that there is some order of cycles of $TTC(N, P')$ such that $S_\kappa(P') = S_\kappa(P)$ for all $\kappa < k$.[3]

Step $t = 1$. For each $j \in S_1(P)$, x_j was top-ranked under P_j. By assumption, x_j is still top-ranked under P'_j. Then under $TTC(N, P')$, the same cycle exists in the graph at step 1, so there is an order of cycles of $TTC(N, P')$ such that $S_1(P') = S_1(P)$.

Step $t < k$. Suppose there is some order of cycles of $TTC(N, P')$ such that $S_\tau(P') = S_\tau(P)$ for all $\tau < t$. Under this order, $N \setminus R_{t-1}(P') = N \setminus R_{t-1}(P)$. By assumption, for every $j \in S_t(P)$, $x_j P'_j w_h$ for all $h \in N \setminus R_{t-1}(P)$ where $w_h \neq x_j$. Thus x_j is top-ranked under P'_j among remaining houses for all $j \in S_t(P)$. Then under this order of $TTC(N, P')$, the cycle $S_t(P)$ also exists in the graph at this step, so there is an order such that $S_t(P') = S_t(P)$.

We have shown that there is some order of cycles of $TTC(N, P')$ such that $S_\kappa(P') = S_\kappa(P)$ for $\kappa < k$. Under this order, $R_{k-1}(P') = R_{k-1}(P)$. Since $i \in S_k(P)$, $x'_i P_i x_i$ implies that $x'_i = w_j$ for some $j \in R_{k-1}(P)$. Therefore, $j \in R_{k-1}(P')$ and $i \in R_{k-1}(P')$. But then $i \in R_{k-1}(P)$, contradicting the assumption that $i \in S_k(P)$. $\square$

We now update the proof of Bird's main theorem using the corrected lemma. The argument proceeds in the same manner as the original.

Theorem 1 (Bird, 1984, Theorem). *TTC is weakly group strategy-proof.*

Proof. Suppose there is a subset $Q \subseteq N$ reporting $P'_Q \neq P_Q$. Let agent $i \in S_k(P)$ be the first agent in Q to enter a trading cycle in $TTC(N, P)$. If there are multiple such agents, i.e. $|S_k(P) \cap Q| \geq 2$, let i be any such agent. We will show that i cannot strictly improve.

Toward a contradiction, suppose that $x'_i P_i x_i$. Note this requires $k \geq 2$, since agents in $S_1(P)$ top-rank x_i. By Lemma 1, there exist $j \in S_\kappa(P)$ and $h \in N \setminus R_{\kappa-1}(P)$, where $\kappa < k$, such that $w_h P'_j x_j$. Then $P'_j \neq P_j$ and $j \in Q$. But then i could not have been the first agent (or one of the first) in Q to enter a trading cycle in $TTC(N, P)$, a contradiction. $\square$

4 Strong group strategy-proofness

Bird (1984) also presents strong group strategy-proofness as a corollary.

Theorem 2 (Bird, 1984, Corollary). *TTC is strongly group strategy-proof.*

He gives the following justification:

> [The corollary] follows directly. Trader j must misrepresent his preferences if trader i is to do better. Since the preferences are strict, trader j forms a cycle and receives a good that he does not prefer to the one he would receive under the original top trading cycle.

[3] The intuition is that the earlier cycles are all the same. But note that the order of cycles in TTC may not be unique; this is the case if multiple cycles are present in the graph at once. This does not substantially affect the intuition, but does require more careful notation.

We feel this requires more elucidation. It is not immediate from strict preferences that j forms a cycle while pointing at the worse house. Moreover, the outcome is produced by a *group* misrepresentation, so there may be other deviating agents apart from j. Thus we provide a proof of strong group strategy-proofness using the key insight from Lemma 1. While other proofs are available,[4] we provide a new one following the ideas laid out here.

We first state the following lemma.

Lemma 2. *Let $x = T(N, P)$ and $Q \subseteq N$. Let P''_Q be such that for all $q \in Q$, P''_q top-ranks x_q and ranks the remaining houses in any order. Denote $P'' = (P''_Q, P_{-Q})$. Then $T(N, P'') = x$.*

That is, if a subset of agents deviate and top-rank the houses they receive in TTC, the resulting TTC allocation is the same. Similar claims are proven in Miyagawa (2002) and Pápai (2000), but we provide a short proof for convenience.

Proof. We show by strong induction on the steps of $TTC(N, P)$ that there is some order of cycles of $TTC(N, P'')$ such that $S_k(P'') = S_k(P)$ for all k.

Step $t = 1$. For each $i \in S_1(P)$, x_i is top-ranked under P_i. It is also top-ranked under P''_i. Therefore, the same cycle exists in the graph at step 1 of $TTC(N, P'')$, so there is an order of cycles of $TTC(N, P'')$ such that $S_1(P'') = S_1(P)$.

Step $t = k$. Suppose that there is an order of cycles of $TTC(N, P'')$ such that $S_t(P'') = S_t(P)$ for all $t < k$. Under this order, $N \setminus R_{k-1}(P'') = N \setminus R_{k-1}(P)$. That is, the remaining agents and houses at step k are the same under either preference profile. In particular, all $i \in S_k(P)$ remain at step k of $TTC(N, P'')$. For each $i \in S_k(P)$, x_i is i's top-ranked house under P_i among $N \setminus R_{k-1}(P)$. For any $i \in Q^C \cap S_k(P)$, since $P''_i = P_i$ and $N \setminus R_{k-1}(P'') = N \setminus R_{k-1}(P)$, x_i is i's top-ranked house under P''_i among remaining houses at step k of $TTC(N, P'')$. For any $i \in Q \cap S_k(P)$, x_i is i's top-ranked house under P''_i and remains at step k of $TTC(N, P'')$. Therefore, each $i \in S_k(P)$ remains at step k of $TTC(N, P'')$ and top-ranks x_i among the remaining houses. As a result, the same cycle exists in the graph at step k of $TTC(N, P'')$, so there is an order of cycles of $TTC(N, P'')$ such that $S_k(P'') = S_k(P)$.

Thus, there exists an order of cycles of $TTC(N, P'')$ such that $S_k(P'') = S_k(P)$ for all k. It follows immediately that $T(N, P'') = T(N, P)$. □

We now prove strong group strategy-proofness by applying Lemmas 1 and 2.

Proof of Theorem 2. Let $Q \subseteq N$ and P'_Q be a misreport. Denote $P' = (P'_Q, P_{-Q})$, $x = T(N, P)$, and $x' = T(N, P')$. Suppose there exists $i \in Q$ such that $x'_i P_i x_i$. We seek to show that some $j \in Q$ is strictly worse off under x'.

Define P''_Q such that for each $q \in Q$, P''_q top-ranks x'_q and preserves the rest of the rankings in P_q. By Lemma 2, $T(N, P'') = x'$, where $P'' = (P''_Q, P_{-Q})$.

Applying Lemma 1 to (N, P'') and x', there exists $j \in S_\kappa(P)$ such that $w_h P''_j x_j$ for some $h \in N \setminus R_{\kappa-1}(P)$. Since $j \in S_\kappa(P)$ and $h \in N \setminus R_{\kappa-1}(P)$, we have $x_j P_j w_h$. Therefore, $P_j \neq P''_j$

[4] Such as Moulin (1995).

and $j \in Q$. The only change from P_j to P''_j is to top-rank x'_j under P''_j, so it must be that $w_h = x'_j$. Thus, $x_j P_j x'_j$ as desired. $\square$

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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