House-Swapping with Commodified Objects*

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Abstract

We study the exchange of indivisible objects ("house-swapping") when the goods may be commodified. In many house-swapping markets, some objects may effectively be indistinguishable from one another, as with dorm rooms or school seats. Thus, all agents are indifferent between copies of the same variety. We call this setting "commodified objects". Top trading cycles (TTC) with fixed tie-breaking has been suggested and used in practice to deal with indifferences in house-swapping problems. However, with general indifferences, TTC with fixed tie-breaking is not Pareto efficient or group strategy-proof. Further, it may not select the strict core, even when it exists. Under commodified goods, agents are *always and only* indifferent between copies of objects. In this setting, TTC with fixed tie-breaking maintains Pareto efficiency, group strategy-proofness, and strict core selection. We also show that in any more general setting, TTC with fixed tie-breaking will not retain any of these properties.

1 Introduction

Important markets such as living donor organ transplants, dorm assignments, and school choice can be modeled as "house-swapping" problems. In a house-swapping problem, each agent is endowed with an indivisible object (called a "house") and has preferences over the set of objects. The objective is to sensibly re-allocate these objects among the agents. Monetary transfers are disallowed, and participants have property rights to their own endowments. **shapley_cores_1974** first introduce house-swapping when agents have strict preferences over houses. The usual stability notion is the core; an allocation is in the core if no subset of agents would prefer to trade their endowments among themselves. Gale's *top trading cycles* (TTC) algorithm finds an allocation in strong core. **roth_weak_1977** further show that the strict core is non-empty, unique, and Pareto efficient. **Roth1982** shows that TTC is strategy-proof; **moulin1995**; **bird1984**; **sandholtztai24**; **papai2000** show it is group strategy-proof. These properties make TTC a normatively attractive algorithm.

The assumption that preferences are strict is quite strong. In particular, if the houses are not unique, agents should naturally be indifferent. We present a model of house-swapping where there are indistinguishable copies of objects ("types"). The model restricts agents to be always indifferent between copies of the

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same object, but never between distinct objects. We call this problem "house-swapping with commodified objects". This models important situations where the house-swapping model is applied in practice. For example, in dorm or public housing assignments, many units are effectively the same (same floor plan in the same building, for example). Likewise in school assignments, different slots at the same school are indistinguishable. We see commodified goods as a minimalist model of indifferences, where indifferences are most plausible (or perhaps undeniable!).

In the fully general setting where agents' preferences may contain any indifferences, *TTC with fixed tie breaking* is often used in practice; ties in preference orders are broken by some external rule. **abdulkadiroglusonmez2003** propose it in the setting of school choice with priorities. However, it is not Pareto efficient or group strategy-proof. Indeed, **ehlers2002coalitional** shows that these two properties are not compatible in house-swapping with indifferences. Additionally, the strict core may be empty or non-unique, and TTC with fixed tie breaking may not select a strict core allocation when one exists.

Commodified goods adds structure to the general case of indifferences by constraining any indifferences to be universal among agents and by limiting the set of preference rankings agents may submit. While the strong core still may not exist, it is essentially unique when it does exist. We show that in house-swapping with commodified goods, TTC with fixed tie-breaking recovers Pareto efficiency and group strategy-proofness. It also selects the unique strong core when it exists, and selects an element in the weak core otherwise. We also show that the commodified goods setting is a maximal setting such that these properties hold, in the sense that allowing a superset of possible preference orderings breaks each property.

In summary, we present a reasonable model of indifferences, commodified goods, which can capture settings where house-swapping is used in practice. Further, there is an advantage in working in the commodified goods setting over the more general setting of full indifferences, as TTC with fixed tie breaking preserves Pareto efficiency, core selection, and group strategyproofness. Thus we also provide a strong normative argument for using TTC with fixed tie-breaking in settings like school choice and dorm assignment.

In addition to the papers already mentioned, our paper contributes to a broader literature on object assignment problems. A number of important papers deal with the object allocation problem without endowments; e.g. **papai2000** and **ehlersklauspapai**. Recently, others have proposed mechanisms for the house swapping model with indifferences; in particular, **quint_houseswapping_2004** and **jaramillo-manjunath**.

Section 2 presents the formal notation. Section 3 explains TTC with fixed tie breaking. Section 4 provides the main results. Section 5 concludes.

2 Model

We present the model primitives. First we recount the classical **shapley_cores_1974** domain. Afterwards we introduce our "commodified objects" domain.

We now present the general house-swapping model (with distinguishable objects). Let $N = \{1, \dots, n\}$ be a finite set of agents, with generic member i. Let $H = \{h_1, \dots, h_n\}$ be a set of houses, with generic member h. Every agent is endowed with one object, given by a bijection $w : N \to H$. The set of all endowments is $W(N, H)$ or W for short. An allocation is an assignment of an object to each agent, given by a bijection $x : N \to H$. The set of all allocations is likewise $X(N, H)$ or X. We denote $x(i) = x_i$ and $w(i) = w_i$ for short.

Each agent has preferences R_i over H. A preference profile is $R = (R_1, R_2, ..., R_n)$. Let $\mathcal{R}_i$ be the set of i's possible preferences. A set $\mathcal{R}_i$ of possible preference orderings is a **domain**. we restrict attention in this paper to preference profiles drawn from $\mathcal{R}_i^N$; that is, every agent has the same set of possible preference orderings. If every $\mathcal{R}_i$ is the set of strict preference orderings, it is the classical **strict preferences domain**. If every $\mathcal{R}_i$ is the set of weak preference orderings, it is the classical **general indifferences domain**.

Our main domain is **commodified objects**. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_K\}$ be a partition of H. Given H and $\mathcal{H}$, denote $\eta : H \to \mathcal{H}$ as the mapping from a house to the partition containing it; that is, $\eta(h) = H_k$ if $h \in H_k$. Each agent i has a strict linear order $\geq_i$ over $\mathcal{H}$, and preferences over H are derived from this. Formally, for $h, h' \in H$,

$$hR_ih' \iff \eta(h) \geq_i \eta(h')$$

The partition $\mathcal{H}$ defines the house types. $\mathcal{R}_i(\mathcal{H})$ is set of preferences given by the partition; we sometimes suppress $(\mathcal{H})$ from the notation when context makes it clear. Given $\mathcal{H}$, $\mathcal{R}(\mathcal{H}) := \mathcal{R}_i(\mathcal{H})^N$ is a commodified objects domain. Note that all agents are indifferent between houses in the same partition and have strict rankings between houses in different partitions. Because of this, we refer to indifference classes for the domain with the understanding that everyone shares the same indifference classes. In this notation, we treat the commodified objects as having identities. I.e. we keep track of the objects in $\mathcal{H}_1$; however, the objects are indistinct and *always* have the same welfare implications.

Commodified objects models settings where some objects are indistinguishable to all participants. An example is dorm rooms or public housing, where there may be many units of the same basic layout and amenities.

2.1 Rules

This subsection recounts formalities on rules (mechanisms) and top trading cycles. Familiar readers may safely skip this subsection.

A market is a tuple (N, H, w, R). A **rule** is a function $f : \mathcal{R} \to X$; given a preference profile, it produces an allocation. When it is unimportant or clear, we suppress inputs from the notation. Denote $f_i(R)$ to be i's allocation; and $f_Q = \{f_i : i \in Q\}$. Fix a rule f and setting. We work with the following desiderata ("axioms").

A rule is Pareto efficient if it always produces Pareto efficient allocations.

Pareto efficiency (PE). For all $R \in \mathcal{R}$, there is no other allocation $x \in X$ such that $x_iR_if_i$ for all $i \in N$ and $x_iP_if_i$ for at least one i.

Strategy-proofness ensures no agent can improve his outcome by submitting false preferences. That is, agents are weakly incentivized to tell the truth.

Strategy-proofness (SP). For all $R \in \mathcal{R}$, for any $q \in N$ and R'_q, $f_q(R)R_qf_q(R'_q, R_{-q})$.

Group strategy-proofness is stronger than SP. It requires that no coalition of agents can improve their outcomes by submitting false preferences. Note that in the following, we require both the true preferences and potential misreported preferences to come from the same set $\mathcal{R}$.

Group strategy-proofness (GSP). For all $R \in \mathcal{R}$, there do not exist $Q \subseteq N$ and R'_Q such that $(R'_Q, R_{-Q}) \in \mathcal{R}$ and $f_q(R'_Q, R_{-Q})R_qf_q(R)$ for all $q \in Q$ with $f_q(R'_Q, R_{-Q})P_qf_q(R)$ for at least one.

Individual rationality models the constraint of voluntary participation. It requires that agents do at least as well as their own endowments.

Individual rationality (IR). For all w and $R \in \mathcal{R}$, $f_i R_i w_i$.

We also define the core, which is a property of allocations. An allocation is in the core if there is no subset of agents who would rather trade their endowments among themselves.

Definition 1. An allocation x is blocked if there exists a coalition $N' \subseteq N$ and allocation x' such that $w(N') = x'(N')$ and for all $i \in N'$, $x'_i R_i x_i$, with $x'_i P_i x_i$ for at least one. An allocation x is in the **core** if it is not blocked.

The weak core is requires that all members of the coalition are strictly better off.

Definition 2. An allocation x is weakly blocked if there exists a coalition $N' \subseteq N$ and allocation x' such that $w(N') = x'(N')$ and for all $i \in N'$, $x'_i P_i x_i$. An allocation x is in the **weak core** if it is not weakly blocked.

The core property models the restriction imposed by property rights. Notice that individual rationality excludes blocking coalitions of size 1. The last axiom is core-selecting.

Core selecting (CS). For all $R \in \mathcal{R}$ and $w \in W$, $f(R)$ is in the core, if the core is nonempty.

We will present results that commodified objects is a largest domain on which $\text{TTC}_\succ$ is PE or CS. It is also "essentially" a largest domain on which $\text{TTC}_\succ$ is GSP (we will note the technicalities when we present the result). By the "largest domain", we mean the following.

Definition 3. A domain $\mathcal{R} = \mathcal{R}_i^N$ is **(symmetric-) maximal** for an axiom A and a rule f if:

1. f is A on $\mathcal{R}_i^N$
2. for any $\tilde{\mathcal{R}}_i \supsetneq \mathcal{R}_i$, f is not A on $\tilde{R}_i^N$
3. (For symmetric-maximal: if $hP_ih' \notin \tilde{\mathcal{R}}_i$ but $hP_ih' \in \tilde{\mathcal{R}}_i$ then also $h'P_ih \in \tilde{\mathcal{R}}_i$.)

Note that this definition of maximality depends on both the domain and the rule f, which differs from elsewhere in the literature. Also note that we restrict to the same set of possible preferences for every agent in both the maximal domain and for any expanded domain, which is a common in the literature.

Symmetric maximality requires that if any indifference is broken, then both strict relations are added to the domain. Of course, symmetric maximality is a weaker condition, as it restricts the possible domain expansions. However, it is natural in a situation requiring preference solicitation to allow either ranking. The third restriction rules out situations where some agents may have a strict ranking only in a particular direction, while others are indifferent.[1] Our results for PE and CS will be with respect to maximality. Our GSP result is with respect to symmetric maximality.

[1]For example, a seat at a school with a scholarship is strictly preferred to a seat to the same school without one, unless a student has outside funds.

3 Top trading cycles with fixed tie breaking

In this paper, we analyze top trading cycles (with tie breaking) in the settings defined in the previous section. For an extensive history, we refer the reader to **MorillRoth24**. We briefly define TTC and TTC with fixed tie-breaking.

Algorithm 1. ***Top Trading Cycles***. *Consider a market* (N, H, w, R) *under strict preferences. Draw a graph with* N *as nodes.*

1. *Draw an arrow from each agent* i *to the owner (endowee) of his favorite remaining object.*

2. *There must exist at least one cycle; select one of them. For each agent in this cycle, give him the object owned by the agent he is pointing at. Remove these agents from the graph.*

3. *If there are remaining agents, repeat from step 1.*

We denote this as $TTC(R)$.

TTC is only well defined with strict preferences, as step 1 requires a unique favorite object. In practice, a **fixed tie breaking** rule is often used to resolve indifferences. Given N, let $\succ = (\succ_1, \ldots, \succ_N)$, where each $\succ_i$ is a strict linear order over N. This linear order will be used to break indifferences between objects (based on their owners). Then let $R_{i,\succ}$ be given by the following. For any $j \neq j'$, let $w_j P_{i,\succ} w_{j'}$ if either

1. $w_j P_i w_{j'}$, or

2. $w_j I_i w_{j'}$ and $j \succ_i j'$

Then $R_{i,\succ}$ is a strict linear order over the individual houses. Example 1 illustrates a tie-break rule. Let $R_\succ = (R_{1,\succ}, \ldots, R_{N,\succ})$. Given a fixed tie breaking rule, **TTC with fixed tie breaking (TTC$_\succ$)** is $TTC_\succ(R) = TTC(R_\succ)$. That is, the tie breaking rule is used to generate strict preferences, and TTC is applied to the resulting profile. Formally, each tie breaking profile $\succ$ generates a different TTC with fixed tie breaking rule.

Example 1. Let $N = \{1, 2, 3, 4\}$.

R_1		$\succ_1$		$R_{1,\succ}$
		1		w_3
w_3, w_4	$+$	2	$\rightarrow$	w_4
w_1, w_2		3		w_1
		4		w_2

3.1 Relation to school choice with priorities

We briefly note that TTC with commodified goods is not identical to TTC with school choice with priorities. Intuitively, the fixed tie-breaking rule determines for i whom to point at; while a school priority determines who points at i. Consider an example with 3 schools and 4 students.

Example 2. Let the set of schools (objects) be $H = \{A, B, C\}$, with C having two slots. Let the students be $N = \{a_1, b_1, c_1, c_2\}$, where a_1 is "endowed" with A, and so on.

Let the school priorities be given by

A	B	C
a	b	c_1
b	a	c_2
c_1	c_2	a
c_1	c_1	b

Alternatively, let a fixed tie breaking rule $\succ$ be given by

a	b	c_1	c_2
c_1	c_2	c_1	c_1
c_2	c_1	c_2	c_2

Finally, compare two alternatives for student preferences

a	b	c_1	c_2
C	C	A	A
A	A	B	B
B	B	C	C

and

a	b	c_1	c_2
C	C	B	B
A	A	A	A
B	B	C	C

TTC with school priorities results in Ac_1, Bc_2, Cab and Ac_2, Bc_1, Cab respectively. Crucially, c_1 gets the preferred school in either case, since it depends on school C's priority. TTC$_\succ$ results in Ac_1, Bc_2, Cab and Ac_1, Bc_2, Cab respectively. Either c_1 or c_2 will get the more preferred school, since it depends on a's or b's tie breaking rule.

4 Results

In general, TTC$_\succ$ is not Pareto efficient, core selecting, nor group strategyproof. However, we show that when preferences are restricted to the commodified objects domain, TTC$_\succ$ satisfies all three properties. Furthermore, commodified objects is maximal on which TTC$_\succ$ is PE and CS, and symmetric-maximal on which it is GSP.

We first illustrate that TTC$_\succ$ is not Pareto efficient in general indifferences. Example 3 gives the simplest case.

Example 3. Let $N = \{1, 2\}$ and preferences be given by $w_1 I_1 w_2, w_1 P_2 w_2$. Let $\succ_i = (1, 2)$ for both agents. In the first round of TTC$_\succ$, both agents point to themselves. The allocation is $x = (w_1, w_2)$, which is Pareto dominated by $x' = (w_2, w_1)$.

The example illustrates the problem with indifferences – TTC$_\succ$ may not take advantage of Pareto gains made possible by the indifferences. The commodified objects domain rules out these situations. Our first main result is that TTC$_\succ$ is Pareto efficient in commodified objects.

Proposition 1. *TTC$_\succ$ is PE in $\mathcal{R}(\mathcal{H})$ for any $\mathcal{H}$ and any tie-breaking profile $\succ$. For any $\mathcal{H}$ and $\succ$, $\mathcal{R}(\mathcal{H})$ is a maximal domain on which TTC$_\succ$ is PE. Further, any other domain on which TTC$_\succ$ is PE for any tie-breaking profile $\succ$ is a subset of some $\mathcal{R}(\mathcal{H})$.*

Proof. Appendix. □

The intuition is that commodified goods rules out situations like in Example 3. In contrast, under commodified goods, agent 1 will have to leave his allocated indifference class in order to benefit agent 2. Any more general domain will reintroduce this possibility.

Our second result deals with the core. In general indifferences, the set of core allocations may not be a singleton. There may be no core allocations or multiple. Likewise, the set of core allocations may be empty or multi-valued in commodified objects, as Example 4 illustrates. However, the core is *essentially unique* when it exists, in that all core allocations are re-arrangements of indistinguishable copies.

Example 4. Let R be given by the following.

R_1	R_2	R_3
w_2, w_3	w_1	w_1
w_1	w_2, w_3	w_2, w_3

It is straight forward to check that the core is empty.

Furthermore, $\text{TTC}_\succ$ always selects the core for any tie-breaking rule $\succ$ in commodified goods. This is in contrast to the result from **ehlers2014** for general indifferences, where only the weak core is guaranteed.

Proposition 2. *Let $x = TTC_\succ(R)$ and $R \in \mathcal{R}(\mathcal{H})$. For any $\succ$,*

1. *x is in the weak core.*
2. *if the core exists, then x is in the core. That is, $TTC_\succ$ is CS.*
3. *if y is in the core, then $x_i I_i y_i$ for all $i \in N$.*

$TTC_\succ$ is CS in $\mathcal{R}(\mathcal{H})$ for any $\mathcal{H}$ and any tie-breaking profile $\succ$. For any $\mathcal{H}$ and $\succ$, $\mathcal{R}(\mathcal{H})$ is a maximal domain on which $TTC_\succ$ is CS. Further, any other domain on which $TTC_\succ$ is CS for any tie-breaking profile $\succ$ is a subset of some $\mathcal{R}(\mathcal{H})$.

Point 3 is the "essential uniqueness" of the core. Since indifferences are universal, it says that all core assignments are rearrangements of copies of object types. Point 2 is implied by 3, but listed separately for clarity. In summary, the $\text{TTC}_\succ$ always produces an allocation in the weak core, produces an allocation in the core when it exists, and the core allocation is unique up to the identities of the commodified objects.

Proof. Appendix. □

The intuition is to that for Pareto efficiency; note any allocation in the core is Pareto efficient. Under general indifferences, the core may be multi-valued due to re-arranging objects that agents are indifferent between. Under commodified goods, this is simply re-arranging copies of indistinguishable objects.

Our third result is that $\text{TTC}_\succ$ is group strategyproof in commodified objects. $\text{TTC}_\succ$ is not GSP in general indifferences. Example 5 illustrates; an agent can break his own indifference to benefit a coalition member without harming himself.

Example 5. Let R and R' be given by the following, and let $Q = \{1, 3\}$.

R_1	R_2	R_3	R'_1
w_2, w_3	w_1	w_1	w_3
w_1	w_2	w_2	w_1, w_2
	w_3	w_3	

Let $\succ_i = (1,2,3)$ for all i. Then $\text{TTC}_\succ(R) = (w_2, w_1, w_3)$. But if 1 misreports R'_1, then $\text{TTC}_\succ(R') = (w_3, w_2, w_1)$. Then 1 is indifferent, and 3 is strictly better off.

Commodified objects eliminates possibilities like the above in a subtle way. The model imposes "exogenous" indifferences; agents can *only* report they are indifferent between all objects in the same indifference class given by $\mathcal{H}$.[2]

Proposition 3. *$TTC_\succ$ is GSP in $\mathcal{R}(\mathcal{H})$ for any $\mathcal{H}$ and any tie-breaking profile $\succ$. For any $\mathcal{H}$ and $\succ$, $\mathcal{R}(\mathcal{H})$ is a symmetric-maximal domain on which $TTC_\succ$ is GSP. Further, any other symmetric domain*[3] *on which $TTC_\succ$ is PE for any tie-breaking profile $\succ$ is a subset of some $\mathcal{R}(\mathcal{H})$.*

Proof. Appendix. □

The proof builds on **moulin1995**, relying on the restriction that agents can only report their indifference class and not arbitrarily break ties within them. This rules out cases like Example 5. A profitably deviating coalition would require a "first mover" misreport in order to obtain a welfare equivalent but *distinct* object (see also the proof in **bird1984**; **sandholtztai24**), which commodified goods rules out. We also note that $\mathcal{R}(\mathcal{H})$ is not a maximal domain on which $\text{TTC}_\succ$ is GSP with the following example.

Example 6. Consider $H = \{h_1, h_2\}$ and $\mathcal{H} = \{\{h_1, h_2\}\}$. Let $\mathcal{R}' = \mathcal{R}(\mathcal{H}) \cup (h_1 P h_2)$. That is, expand the domain by including the ordering $h_1 P h_2$. It can be verified that $\text{TTC}_\succ$ is still group strategyproof.

If $R_1 = R_2 = (1P1)$ or $R_1 = R_2 = (1I2)$, then of course there is no possible group manipulation. Now let $h_1 = w_1$ and $h_2 = w_2$, and $1 \succ_i 2$ for both i. Consider two possible (true) preference profiles:

R_1	R_2		R_1	R_2
h_1	h_1, h_2	or	h_1, h_2	h_1
h_2				h_2

In the first case, there is no improving allocation. In the second case, it would be advantageous for agent 1 to claim h_2 and pass along h_1, but this is not possible, since this preference is not in $\mathcal{R}'$. Now let $h_1 = w_2$ and $h_2 = w_1$. In the first case, there is again no improving allocation (they trade in $\text{TTC}_\succ$). In the second case, there is again no improving allocation. It can also be verified that no other tie breaking rule $\succ$ allows an improving coalition.

The following theorem collects the results presented above.

Theorem 1. *$TTC_\succ$ is PE, CS, and GSP for all tie-breaking profiles $\succ$ in any $\mathcal{R}(\mathcal{H})$. Each $\mathcal{R}(\mathcal{H})$ is a maximal domain on which $TTC_\succ$ is PE and CS. Each $\mathcal{R}(\mathcal{H})$ is a symmetric-maximal domain on which $TTC_\succ$ is GSP. Additionally, if the core of (N, w, R) for $R \in \mathcal{R}(\mathcal{H})$, it is unique up to the identity of the objects.*

The results and examples suggest a practical issue – selection of $\mathcal{H}$ given the set of objects H. In some cases the commodification may be obvious; e.g., identical tasks or slots in a program. In other cases, there may be some ambiguity; e.g., are two dorms of the same floor plan but on different floors equivalent? Inappropriately combining two indifference classes can lead to efficiency losses in the spirit of Example 3. On the other hand, splitting an indifference class can allow group manipulations like in Example 5. We leave formal results on the tradeoff as future work.

[2]Constraining the reports is also an important difference from **ehlers2002coalitional**.

[3]That is, if the domain contains a preference where hP_ih', then $h'P_ih$ is also in the domain.

4.1 Maximality for GSP

While commodified goods domains are not maximal for $\text{TTC}_{\succ}$ and GSP, we show that a closely related domain is maximal. We will define a domain that is a generalization of commodified goods, but that we feel has no intuitive use. Our intention is to argue informally that commodified goods domains are "essentially" maximal for $\text{TTC}_{\succ}$ and GSP, in that the true maximal domain is not of modeling value.

Given a partition $\mathcal{H} = \{H_1, ..., H_K\}$, let a function $* : \mathcal{H} \to H$ pick one house from each partition, e.g. $H_k := \{h_k^*\} \cup \underline{H}_k$. That is, let every subset be partitioned again into a singleton and the remainder. Let $\bar{R}_i^1(\geq_i, \mathcal{H}, *)$ be given by the following. (We will suppress the dependence on $(\geq_i, \mathcal{H})$). As before, let there be a strict linear order $\geq$ over $\mathcal{H}$. If $h, h' \notin \min_i \mathcal{H}$, let the preference be as before: $h\bar{R}_i^1 h' \iff \eta(h) \geq_i \eta(h')$. If $h \notin \min_i \mathcal{H}, h' \in \min_i \mathcal{H}$, then $h\bar{P}_i^1 h'$. Finally, let $\bar{R}_i^1$ in $\min_i \mathcal{H} := H_k = \{h_k^*\} \cup \underline{H}_k$ be given by the following:

$$h_k^* \bar{P}_i^1 h \text{if } h \in \underline{H}_k$$
$$h \bar{I}_i^1 h' \text{if } h, h' \in \underline{H}_k$$

That is, $\bar{R}_i^1(\geq_i, \mathcal{H}, *)$ is commodified goods *until* the last indifference class; in the last indifference class, h_k^* is strictly preferred to the rest of the indifference class. The following example illustrates.

Example 7. Let $\mathcal{H} = \{\{a, b\}, \{c, d, e\}\}$. Given $\mathcal{H}$, rankings in $\bar{R}_i^1(\geq_i, \mathcal{H}, *)$ reflect both orderings $\geq_i$, and selection of h_k^*. Let $*(\{a, b\}) = a$ and $*(\{c, d, e\}) = c$. Then

$\cup_{\geq_i, *} \left\{\bar{R}_i^1(\geq_i, \mathcal{H}, *)\right\}$	
a, b	c, d, e
c	a
d, e	b

Similarly, define $\bar{R}_i^2(\geq_i, \mathcal{H}, *)$ in the same way, except in $\min_i \mathcal{H} := H_k = \{h_k^*\} \cup \underline{H}_k$ let it be given by the following:

$$h \bar{P}_i^2 h_k^* \text{if } h \in \underline{H}_k$$
$$h \bar{I}_i^2 h' \text{if } h, h' \in \underline{H}_k$$

That is, h_k^* is strictly dispreferred to the rest of the indifference class.

Now denote

$$\bar{\mathcal{R}}_i(\mathcal{H}, *, \text{int}) = \begin{cases} \mathcal{R}_i(\mathcal{H}) \cup \left\{\cup_{\geq_i} \bar{R}_i^1(\geq_i, \mathcal{H}, *)\right\} & \text{if int} = 1 \\ \mathcal{R}_i(\mathcal{H}) \cup \left\{\cup_{\geq_i} \bar{R}_i^2(\geq_i, \mathcal{H}, *)\right\} & \text{if int} = 2 \end{cases}$$

That is, start with commodified goods. Fix a choice of $*$, then append *either* an alternative ranking of the first type or second type.

Example 8. Let $\mathcal{H}$ and $*$ be defined as in Example 7. Then $\bar{\mathcal{R}}_i(\mathcal{H}, *, 1)$ contains the following rankings:

$\bar{\mathcal{R}}_i(\mathcal{H}, *, 1)$			
a, b	a, b	c, d, e	c, d, e
c	c, d, e	a, b	a
d, e			b

We show that $\bar{\mathcal{R}}_i^N$ is a maximal domain for group strategyproofness.

Proposition 4. *$TTC_\succ$ is GSP in $\bar{\mathcal{R}}_i(\mathcal{H})^N$ for any $\mathcal{H}$ and any tie-breaking profile $\succ$. For any $\mathcal{H}$ and $\succ$, $\bar{\mathcal{R}}_i(\mathcal{H})^N$ is a maximal domain on which $TTC_\succ$ is GSP. Further, $\bar{\mathcal{R}}_i(\mathcal{H})$ describe all supersets of $\mathcal{R}_i(\mathcal{H})$ such that $TTC_\succ$ is GSP on $\bar{\mathcal{R}}_i(\mathcal{H})^N$.*

Proof. Appendix. □

While we do not have formal results, we conjecture that $\bar{\mathcal{R}}_i(\mathcal{H})^N$ is the only generalization of commodified goods that is a maximal domain on which $\text{TTC}_\succ$ is GSP for all $\succ$. The proof shows that adding any strict relation above the last indifference class or adding an indifference anywhere breaks GSP.

5 Conclusion

The house-swapping market is a classic model in economic theory with applications to important markets like housing assignment, school choice, and organ exchange. Surprisingly, it took about thirty years from **shapley_cores_1974** to generalize results to indifferences. Since then, there has been a significant amount of work dealing with indifferences.

TTC with fixed tie-breaking is a commonly used mechanism for house-swapping problems with indifferences. Unfortunately, it does not preserve Pareto efficiency, group strategyproofness, or core selection in general indifferences.

We have proposed a model of a particular kind of indifferences, "commodified objects", where there are indistinguishable copies of objects. Commodified objects captures many of the situations where house-swapping is relevant. (Consider for example housing assignment with many indistinguishable dorm rooms.) Therefore it is a compelling case to include in a model of house-swapping.

Fortunately, TTC with fixed tie-breaking preserves the aforementioned properties – Pareto efficiency, group strategyproofness, and core selection – on commodified objects. Moreover, Pareto efficiency and core selection fail on any larger domains. While group strategyproofness is preserved on some larger domains, it fails on any "symmetrically larger" domain. Thus commodified objects is not only a compelling case to include, but also the most general case preserving these properties.

We leave a characterization of TTC with fixed tie-breaking on commodified goods also remains an open question.

Appendix

The appendix contains the proofs of the results in the main text. Throughout, for a partition $\mathcal{H}$, let $\eta : H \to \mathcal{H}$ associate an object h with its indifference class under $\mathcal{R}(\mathcal{H})$. Additionally, given a market and $\text{TTC}_{\succ}(R)$, denote $S_k(R)$ as the kth cycle executed in $\text{TTC}_{\succ}(R)$.[4]

It is immediate that $\text{TTC}_{\succ}$ is IR, as any agent pointing at his own endowment must be assigned to it. We will use this fact for the some of the proofs.

Pareto efficiency

Proposition 1. *$TTC_{\succ}$ is PE in $\mathcal{R}(\mathcal{H})$ for any $\mathcal{H}$ and any tie-breaking profile $\succ$. For any $\mathcal{H}$ and $\succ$, $\mathcal{R}(\mathcal{H})$ is a maximal domain on which $TTC_{\succ}$ is PE. Further, any other domain on which $TTC_{\succ}$ is PE for any tie-breaking profile $\succ$ is a subset of some $\mathcal{R}(\mathcal{H})$.*

Proof. The result is trivial for $|N| = 1$. Now let $|N| \geq 2$.

We show that PE is satisfied on commodified goods. Consider any (N, H, w). Let $\mathcal{H}$ be any partition and let $R \in \mathcal{R}(\mathcal{H})$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose it the partition has at least two subsets. Let $x = TTC_{\succ}(R)$, and suppose $y \in X$ Pareto dominates x. Let $W = \{i : y_i P_i x_i\}$ be the set of agents who strictly improve under y, which must be nonempty. Let $i \in W$ be the first agent in W assigned in $TTC_{\succ}(R)$.

Denote $i \in S_k(R)$ and $\eta(y_i) = H_y$. We have that $y_i P_i x_i$. . Note that at step k, no objects in H_y were available, otherwise i would have pointed to one of them rather than at x_i.

Since $y_i P_i x_i$, we have $x_i \notin H_y$ but $y_i \in H_y$. Thus there must be another agent j such that $x_j \in H_y$ but $y_j \notin H_y$. Since y Pareto dominates x, $y_j R_j x_j$. Since y_j and x_j are not in the same indifference class, we have $y_j P_j x_j$. (This is where commodified objects is used.)

Then $j \in W$. Further, j must have been assigned before step k, since no object in H_y was available at step k. This contradicts the presumption that i was the first agent in W assigned.

We now turn to the maximality and "uniqueness" claims. It suffices to show the uniqueness claim: that $\text{TTC}_{\succ}$ fails PE on any domain $\tilde{\mathcal{R}}$ that is not a subset of some $\mathcal{R}(\mathcal{H})$. If $\tilde{R} \not\subseteq \mathcal{R}(\mathcal{H})$, then it must contain two orderings R_*, R_{**} such that for $h_1, h_2 \in H$ we have $h_1 I_* h_2$ but $h_1 P_{**} h_2$. (Note that R_*, R_{**} are preference orderings, not preference profiles.)

Taking only R_*, R_{**} for granted, we find R, w, and $\succ$ such that $\text{TTC}_{\succ}(R)$ does not produce a Pareto efficient allocation. Let $w_i = h_i$. Let $R_1 = R_*$, so that $h_1 I_1 h_2$. Let $R_2 = R_{**}$, so that $h_1 P_2 h_2$. Now for$i \notin \{1, 2\}$, let R_i be given by:

- if $w_i R_{**} w_1$, then $R_i = R_{**}$;
- otherwise $R_i = R_*$.

Let $\succ$ be such that $i \succ_i j$ for all $i, j \in N$.

This construction will allow us to ignore $i \notin \{1, 2\}$. If $w_i \in \{h : h R_2 w_1\}$, then $w_i R_i w_1$ by construction. Likewise if $w_i \in \{h : h R_1 w_1\}$ then $w_i R_i w_1$; either $R_i = R_{**}$ or $R_i = R_1$. Then if $w_i R_1 w_1$ or $w_i R_2 w_1$ for $i \notin \{1, 2\}$, we have $w_i P_{i,\succ} w_1$. Thus in $\text{TTC}_{\succ}(R) := \text{TTC}(R_{\succ})$, IR applied to $i \notin \{1, 2\}$ requires that $w_1 R_1 x_1$ and $w_1 R_2 x_2$.

[4]Note that S_k may not be unique, since multiple cycles may appear in step 2 of Algorithm 1.

Then at some point, $\text{TTC}_{\succ}(R)$ reaches a step where 1 and 2 are remaining (perhaps among others), but any object either of them weakly prefers to w_1 is already assigned. At this step, 1 points to himself, so $x_1 = w_1$ and $w_1 P_2 x_2$. This is Pareto dominated by $x'_1 = w_2, x'_2 = w_1$, and $x'_i = x_i$ for the rest. □

Core selecting

Proposition 2. *Let $x = TTC_{\succ}(R)$ and $R \in \mathcal{R}(\mathcal{H})$. For any $\succ$,*

1. *x is in the weak core.*
2. *if the core exists, then x is in the core. That is, $TTC_{\succ}$ is CS.*
3. *if y is in the core, then $x_i I_i y_i$ for all $i \in N$.*

$TTC_{\succ}$ is CS in $\mathcal{R}(\mathcal{H})$ for any $\mathcal{H}$ and any tie-breaking profile $\succ$. For any $\mathcal{H}$ and $\succ$, $\mathcal{R}(\mathcal{H})$ is a maximal domain on which $TTC_{\succ}$ is CS. Further, any other domain on which $TTC_{\succ}$ is CS for any tie-breaking profile $\succ$ is a subset of some $\mathcal{R}(\mathcal{H})$.

Proof. We first note a fact about $x = \text{TTC}_{\succ}(R)$. If $i \in S_\ell(R)$ and $h P_i x_i$, then h was assigned in a cycle before ℓ. This follows from the definitions; $h P_i x_i$ implies $h P_{i,\succ} x_i$, and $\text{TTC}_{\succ}(R) = \text{TTC}(R_{\succ})$. Under $\text{TTC}(R_{\succ})$, an object $h P_{i,\succ} x_i$ must have been assigned earlier than ℓ, otherwise i would have pointed to it.

(1.) Let $R \in \mathcal{R}(\mathcal{H})$ and denote $x = \text{TTC}_{\succ}(R)$. Suppose there is a weakly blocking coalition $N' \subseteq N$ with allocation y such that $y_i P_i x_i$ for all $i \in N$. We show by induction on the cycles of $\text{TTC}_{\succ}(R)$ that N is empty.

Step 1. All $i \in S_1(R)$ received one of their top-ranked objects, so they cannot be in N'.

Step k. Suppose N' does not include any members of earlier cycles. Now consider $i \in S_k(R)$. If $y_i P_i x_i$, then y_i must be an object assigned in $\cup_{\ell=1}^{k-1} S_\ell(R)$. But no agents in $\cup_{\ell=1}^{k-1} S_\ell(R)$ are in N', so it is not feasible to include i in N' either. Thus no agents in $S_k(R)$ are in N'.

Then N' is empty, completing the proof of this claim.

(2.) 2 is implied by 3.

(3.) Suppose the core of (N, H, w, R) is nonempty and contains y. Denote $x = \text{TTC}_{\succ}(R)$. We show $x_i I_i y_i$ $(*)$ for all i by induction on the cycles.

Step 1. All $i \in S_1(R)$ received one of their top-ranked objects, so $x_i R_i y_i$. Suppose $(*)$ is not true for $S_1(R)$. Then there is some $i \in S_1(R)$ such that $x_i P_i y_i$. But then $S_1(R)$ and x block against y, a contradiction.

Step k. Suppose that $(*)$ is true for all cycles before k. Suppose for some $i \in S_k(R)$ we have $y_i P_i x_i$. Then y_i was assigned in a cycle before k. Further, y_i and x_i are in different indifference classes. Thus under y if y_i is assigned to i, an agent j in $(\cap_{\ell=1}^{k-1} S_\ell(R)) \cap H_{y_i}$ must be assigned an object outside of H_{y_i}. But then it cannot be that $y_j I_j x_j$, a contradiction.[5] Thus we have that $x_i R_i y_i$ for all $i \in S_k(R)$. Suppose $(*)$ is not true for $S_k(R)$. Then there is some $i \in S_k(R)$ such that $x_i P_i y_i$. But then $S_k(R)$ and x block against y, a contradiction.

[5] This is where commodified goods is used – this claim fails in general indifferences.

Then $x_i I_i y_i$ for all i, as desired.

We now turn to the uniqueness claim: that $\text{TTC}_\succ$ fails CS on any domain $\tilde{\mathcal{R}}$ that is not a subset of some $\mathcal{R}(\mathcal{H})$. We use a similar construction as in the previous proof. If $\tilde{R} \not\subseteq \mathcal{R}(\mathcal{H})$, then it must contain two orderings R_*, R_{**} such that for $h_1, h_2 \in H$ we have $h_1 I_* h_2$ but $h_1 P_{**} h_2$. Without loss of generality, let h_1 be a best-ranked object according to R_{**} such that this is true. That is, $h_1 \in \max_{R_{**}} \{h : \exists h' \text{ such that } h I_* h'\}$.

Let $w_i = h_i$. Let R_i be given by

- if $w_i R_* w_1$ and $i \neq 2$, then $R_i = R_*$. Denote this set of agents as A; note that $1 \in A$ and $2 \notin A$.
- otherwise, $R_i = R_{**}$. Denote this set of agents as B.

Let each $\succ_i$ prioritize himself; that is, $i \succ_i j$ for all $i, j \in N$. We prove that $\text{TTC}_\succ(R) = w$.

Although $\text{TTC}_\succ(R)$ does not set an order in which to execute cycles if multiple are present, we will execute them in the following particular order without loss of generality. We proceed in descending order of indifference classes of R_*. If there exists w_i such that $w_i P_* w_1$, A surely points within themselves, since their top-ranked objects must be held by members of A. By construction of $\succ$, holders of the top-ranked objects point at themselves and receive their own endowments. This continues until only $B \cup \{i \in A : w_1 I w_i\}$ remain. All remaining $i \in A$ point at themselves and are assigned their endowments. Now all remaining agents are in B so have the same preferences. This must lead to $\text{TTC}_\succ(R) = w$ as desired.

However, w is blocked by $x' = (w_2, w_1, w_3, ..., w_N)$. It remains to show that x' is in the core. We prove this by induction on indifference classes of R_*, then on R_{**}. Suppose there is a coalition Q and allocation x'' where all $q \in Q$ receive $x''_q R_q x'_q$. Note that $x'_i = x_i$ for all $i \notin \{1, 2\}$.

Consider the best indifference class of R_*, denoted $T^1 = \arg\max_H R_*$. If $w_1 \notin T_*$, all i such that $w_i \in T^1$ are in A, and $x'_i = w_i$. In order to be included in a coalition Q, we require $x'_i I_i x_i$. Only rearrangements among T^1 are possible, and there are no strict improvements. Further, objects in T^1 are not available for later tranches.

We repeat this argument until we reach the tranche where only $B \cup \{i \in A : w_1 I w_i\}$ remain. That is, $H \backslash \left\{\cup_{t=1}^{k-1} T^t\right\}$ remain. Denote the best remaining indifference class of R_* among $H \backslash \left\{\cup_{t=1}^{k-1} T^t\right\}$ as T^k. Consider i such that $w_i \in T^k$; note that these are the remainder of A and 2; denote this set N^k. Among these, we have $x'_1 = w_2, x'_2 = w_1$, and $x'_i = w_i$ for the remainder of $i \in A$. For the $i \in A$, x'_i are the best available objects, so as before we require $x'_i I_i x_i$. For agent 2, by construction w_2 is the best object among T^k. Thus if any N^k are included in Q, only rearrangements among T^k are possible, and there are no strict improvements.

Now only agents in B remain. They all have the same preferences, so the proceeding steps are immediate. Thus it is impossible for any Q to strictly improve without harming a member, and x' is in the core. □

Group strategyproofness

We now present the proof that $\text{TTC}_\succ$ is GSP in commodified objects. The proof follows **moulin1995** (Lemma 3.3), which proves GSP for strict preferences. It is illustrative to see where the restriction on commodified goods is used to preserve the logic.

Proposition 3. *$TTC_\succ$ is GSP in $\mathcal{R}(\mathcal{H})$ for any $\mathcal{H}$ and any tie-breaking profile $\succ$. For any $\mathcal{H}$ and $\succ$, $\mathcal{R}(\mathcal{H})$ is a symmetric-maximal domain on which $TTC_\succ$ is GSP. Further, any other symmetric domain on which $TTC_\succ$ is PE for any tie-breaking profile $\succ$ is a subset of some $\mathcal{R}(\mathcal{H})$.*

Proof. Let $R \in \mathcal{R}(\mathcal{H})$, and denote $x = \text{TTC}_{\succ}(R)$. Let $S_t(R)$ be the cycles of $\text{TTC}_{\succ}(R)$. Suppose $Q \subseteq N$ is a coalition reporting $R'_Q \in \mathcal{R}(\mathcal{H})$, and denote $x' = \text{TTC}_{\succ}(R'_Q, R_{-Q})$. Suppose that $x'_i R_i x_i$ for all $i \in Q$. The proof is by induction on the cycles of $\text{TTC}_{\succ}(R)$ containing Q.

Let t^* be the smallest index such that $S_{t^*}(R) \cap Q$ is nonempty; that is, the first cycle where member(s) of Q are assigned. By definition of $\text{TTC}_{\succ}$, for all $j \in \cup_{\ell=1}^{t^*-1} S_\ell(R)$, $x_j = x'_j$. For $i \in S_{t^1}(R) \cap Q$, the best possible objects are those in $\eta(x_i)$. Since $x'_i R_i x_i$, under any misreport it must be that $x'_i \in \eta(x_i)$ for all $i \in S_{t^*}(R) \cap Q$. (This is where commodified goods is applied. This step fails under general indifferences, as i could report and obtain a welfare equivalent but distinct object.)

We argue that the same cycle as in $S_{t^*}(R)$ must form under $\text{TTC}_{\succ}(R')$. Note that in $\text{TTC}_{\succ}(R)$, $i \in S_{t^*}(R) \cap Q$ reported $\eta(x_i)$ as the favorite remaining object and pointed at x_i via the tie breaking rule $\succ$. At step t^* of $TTC_{\succ}(R')$, if i pointed elsewhere than $\eta(x_i)$, he did not receive that object, so must not have formed a cycle. In short, i will point uselessly until he points at $\eta(x_i)$. Meanwhile, $j \in S_{t^*}(R) \cap Q^c$ pointed at their original x_j and did not have formed a cycle. The owners of x_j were also in $S_{t^*}(R)$, so j cannot form a cycle until at least one member of $S_{t^*}(R)$ is assigned. Then $i \in S_{t^*}(R) \cap Q$ eventually point at $\eta(x_i)$. Since none of $S_{t^*}(R) \cap Q^c$ were assigned, x_i is still available; i must point to it via $\succ$. Then $S_{t^*}(R)$ still forms (though not necessarily at step t^*).

Now consider the step t^*+1. The same objects are available for assignment to $S_{t^*+1}(R)$ under $\text{TTC}_{\succ}(R')$, and we can repeat the same argument.

We now turn to the symmetric-maximality and uniqueness claims. As before, we show that that $\text{TTC}_{\succ}$ fails GSP on any symmetric domain $\tilde{\mathcal{R}}$ that is not a subset of some $\mathcal{R}(\mathcal{H})$. If $\tilde{R} \not\subseteq \mathcal{R}(\mathcal{H})$, then it must contain two orderings R_*, R_{**} such that for $h_1, h_2 \in H$ we have $h_1 I_* h_2$ but $h_1 P_{**} h_2$. The symmetric requirement also necessitates that it contains R_{***} such that $h_2 P_{***} h_1$. Taking only R_*, R_{**}, R_{***} for granted, we find R, w, and $\succ$ such that $\text{TTC}_{\succ}$ is not group strategyproof.

Let $w_i = h_i$ for all $i \in N$. Let $R_1 = R_*$ and $R_2 = R_{**}$; that is $w_1 I_1 w_2$ and $w_1 P_2 w_2$. Let the rest of R be given by:

1. Let $w_i R_* w_1$, then $R_i = R_*$. Let $A = \{i : R_i = R_*\}$; note this includes 1.

2. If $w_1 P_* w_i$ and $w_i R_{**} w_1$, then $R_i = R_{**}$. Let $B = \{i : R_i = R_{**}\}$; note this includes 2.

3. Otherwise, $R_i = R_{***}$.. Let $C = \{i : R_i = R_{***}\}$.

Let $1 \succ_1 \cdots$ and $1 \succ_2$, and for all other agents $i \succ_i$. We show that $\text{TTC}_{\succ}(R) = w$. By the same argument as in the proof of Proposition 2, the all $i \in A$ receive $x_i = w_i$. Now consider the second group. A similar argument again holds. Suppose $j \in B$. Since w_1 belongs to A, if $w_i R_j w_j$, then either $i \in A$ or $i \in B$. However, all $w_i = x_i$ for $i \in A$. Then we again have $w_i = x_i$ for $i \in B$. Then all remaining agents are in C, so we have $x_i = w_i$ as desired.

We have $w_1 P_2 w_2$ but $w_1 I_1 w_2$. Now suppose 1 instead reports $R'_1 = R_{***}$. Then $A \backslash \{1\}$ all receive $w_i = x'_i$. Since no $i \in B \backslash \{2\}$ strictly prefers $w_1 P_i w_i$, they receive again $w_i = x'_i$. Then 2 is not assigned until all other agents in B are assigned, at which point he points at 1.

Moreover, observe that for all $j \in C$ and any $i \neq j$ such that $w_i P_j w_j$, we have $w_i P_i w_j$. This is because A and B favor their own endowments to w_1, and C favor h_2 to h_1. Consequently, $x_j = w_j$ for all$j \in C$. Therefore, 1 cannot be assigned under$\text{TTC}_{\succ}(R')$ until after all agents in C form self-cycles, at which point 1 points at 2. Then $x'_1 = w_2, x'_2 = w_1$, benefiting 2 without harming 1.

□

We now turn to the results for GSP in the expanded domain $\bar{\mathcal{R}}(\mathcal{H})$. First, we note that TTC is (Maskin) monotone.

Let $L(x_i, R_i) = \{h \in H : x_i R_i h\}$ be the lower contour set of R_i at x_i. The definition is standard; a rule f is monotone if when any set of agents move up their allocations in their rankings, the allocation remains the same.

Monotonicity (MON). Let $x = f(R)$. For all $R \in \mathcal{R}$, if $R' \in \mathcal{R}$ is such that for all $i \in N$ we have $L(x_i, R_i) \subseteq L(x_i, R'_i)$, then $x = f(R')$.

TTC is monotone in strict preferences; we will apply it here with respect to strict tie-broken preferences $R_\succ$.

The following result is adapted from **sandholtztai24**, who show it for TTC with strict preferences. We simply apply it here to $\text{TTC}_\succ(R) \equiv \text{TTC}(R_\succ)$, noting that $R_\succ$ is always a profile of strict preferences.

Lemma 1 (**sandholtztai24**)**.** *Let R, R' be profiles of any preferences. Let $x = TTC(R_\succ)$ and $x' = TTC(R'_\succ)$. Suppose there is some i such that $x'_i P_{i,\succ} x_i$. Then there exists some j such that $w_k P'_{j,\succ} x_j$ and $x_j P_{j,\succ} w_k$.*

Proposition 4. *$TTC_\succ$ is GSP in $\bar{\mathcal{R}}_i(\mathcal{H})^N$for any $\mathcal{H}$ and any tie-breaking profile $\succ$. For any $\mathcal{H}$ and $\succ$, $\bar{\mathcal{R}}_i(\mathcal{H})^N$ is a maximal domain on which $TTC_\succ$ is GSP. Further, $\bar{\mathcal{R}}_i(\mathcal{H})$ describe all supersets of $\mathcal{R}_i(\mathcal{H})$ such that $TTC_\succ$ is GSP on $\bar{\mathcal{R}}_i(\mathcal{H})^N$.*

Proof. Let $R \in \bar{\mathcal{R}}(\mathcal{H})$, and denote $x = \text{TTC}_\succ(R)$. Let $S_t(R)$ be the cycles of $\text{TTC}_\succ(R)$. Suppose $Q \subseteq N$ is a coalition where each $q \in Q$ reports $R'_q \in \bar{\mathcal{R}}(\mathcal{H})$, and denote $x' = \text{TTC}_\succ(R'_Q, R_{-Q})$.

Let t^* be the index of the first cycle in which a coalition member $i \in Q$ is assigned to an object in his last indifference class. We apply the same reasoning as the proof of Proposition 3 to argue that for all $j \in \left(\cup_{t=1}^{t^*-1} S_t(R)\right) \cap Q$ we have $x'_j = x_j$.

Now consider $S_{t^*}(R)$. Only objects in $\eta(x_i)$ remain, since i would have pointed elsewhere if not. In the following, we now focus attention only on remaining agents and rankings over remaining objects. Denote $\eta(x_i) := \{h^*\} \cup \underline{H}$. Agents can either have the standard ranking over $\eta(x_i)$, denoted R_* which is indifferent over $\eta(x_i)$; or the alternative ranking R_{**}. The alternative ranking is either 1) $h^* P_{**} h$ for all $h \in \underline{H}$, which we denote as $h^* P_{**} \underline{H}$; or 2) $\underline{H} P_{**} h^*$. (Note that in either case, R_{**} is indifferent over $\underline{H}$.) We treat these two cases separately.

Toward a contradiction, suppose there is some $q \in Q$ such that $x'_q P_q x_q$. We show some other agent in Q is strictly worse off.

1. Suppose the alternative ranking is $h^* P_{**} \underline{H}$. The strict improver must have $x'_q = h^*$, $x_q \in \underline{H}$, and $h^* P_q \underline{H}$.

   Define a new preference profile R'' as $h^* P''_q \underline{H}$ and $h^* I''_i \underline{H}$ for $i \neq q$. Let $\text{TTC}(R''_\succ) = x''$.

   For $i \neq q$, we have $L(x'_i, R'_{i,\succ}) \subseteq L(x'_i, R''_{i,\succ})$, since the only possible change was to add h^*. For q, $L(x'_q, R'_{q,\succ}) = L(x'_q, R''_{q,\succ})$, since $x'_q = h^*$ and all objects are weakly dispreferred to h^* under any preference. By monotonicity of TTC, $x'' = x'$. In particular, $x''_q = h^*$ and $x''_q P_q x_q$.

We can apply Lemma 1 to x'' and x. There must exist some $j \in Q$ and object h such that $x_j P_{j,\succ} h$ and $h P''_{j,\succ} x_j$. But the only possible relative change in rankings involves h^*. So either $h = h^*$ or $x_j = h^*$.

If $h = h^*$, then the relative ranking of h^* moved up from $P_{j,\succ}$ to $P''_{j,\succ}$. It must be that $h^* I_j \underline{H}$ and $h^* P''_j \underline{H}$. By construction of R'', this implies $j = q$, but this contradicts $h^* P_q \underline{H}$.

Then it must be $x_j = h^*$. Likewise, the relative ranking of h^* moved down from $P_{j,\succ}$ to $P''_{j,\succ}$, so it must be that $h^* P_j \underline{H}$ and $h^* I''_j \underline{H}$. But then $x_j P_j x'_j$ as desired.

2. Suppose the alternative ranking is $\underline{H} P_{**} h^*$. Then the strict improver must have $x'_q \in \underline{H}$, $x_q = h^*$, and $\underline{H} P_q h^*$.

   Let ℓ be the agent such that $x'_\ell = h^*$. Since $x_q = h^*$, $x_\ell \in \underline{H}$. Define a new preference profile R'' as $h^* I''_\ell \underline{H}$ and $\underline{H} P''_i h^*$ for $i \neq \ell$. Let $\text{TTC}(R''_\succ) = x''$.

   For $i \neq \ell$ we have $L(x'_i, R'_{i,\succ}) \subseteq L(x'_i, R''_{i,\succ})$, since the only possible change was to add h^* to the lower contour set. For ℓ, $L(x'_\ell, R'_{\ell,\succ}) = L(x'_\ell, R''_{\ell,\succ})$, since $h^* I''_\ell \underline{H}$. By monotonicity of TTC, $x'' = x'$. In particular, $x''_q \in \underline{H}$ and $x''_q P_q x_q$.

   As before, we apply Lemma 1 to x'' and x. There must exist some $j \in Q$ and object h such that $x_j P_{j,\succ} h$ and $h P''_{j,\succ} x_j$. Either $h = h^*$ or $x_j = h^*$.

   If $x_j = h^*$, then $j = q$ is the strict improver. (Note there can only be one strict improver since h^* is unique.) But $\underline{H} P_q h^*$ and since $q \neq \ell$, $\underline{H} P''_q h^*$, but this contradicts the requirement that $x_j P_{j,\succ} h$ and $h P''_{j,\succ} x_j$.

   Then it must be $h = h^*$. We require $h^* P''_{j,\succ} x_j$. But $\underline{H} P''_i h^*$ for $i \neq \ell$, so it must be $j = \ell$. Since we have $h^* I''_\ell \underline{H}$ and $h^* P''_{\ell,\succ} x_\ell$ but $x_\ell P_{\ell,\succ} h$, we must have $\underline{H} P_\ell h^*$. Since $x_{\ell'} = h^*$, this gives us $x_\ell P_\ell x'_\ell$, as desired.

We now prove the maximality and uniqueness claims. Let $\tilde{\mathcal{R}}_i \supsetneq \bar{\mathcal{R}}_i$ and denote $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_i^N$. We will show there exists a w, $R \in \tilde{\mathcal{R}}$, and $\succ$ such that $\text{TTC}_\succ(R)$ is not GSP. At least one of the following must be true:

I There is $R_* \in \tilde{\mathcal{R}}_i$ but $R_* \notin \bar{\mathcal{R}}_i(\mathcal{H})$ for any $\mathcal{H}$.

II There are $R_*, R_{**} \in \tilde{\mathcal{R}}_i$ but there is no $\mathcal{H}$ such that $R_*, R_{**} \in \bar{\mathcal{R}}_i(\mathcal{H})$. That is, perhaps both R_* and R_{**} are from commodified goods-plus domains, but they cannot be from the same one.

Suppose case I is true. There must be some $R_i, R_j \in \tilde{\mathcal{R}}_i$ such that at least one of the following occurs:

1. there exist $a, b \in H$ such that $\eta(a) \neq \eta(b)$ but $a I_i b$. That is, an extra indifference is present.

2. there exist $a, b, c \in H$ such that $\eta(a) = \eta(b) \neq \eta(c)$ but $a P_i b$ and $a P_i c$. That is, an indifference outside of the last indifference class is broken (in any way).

3. there exist $a, b, c, d \in H$ such that $a, b, c, d \in \eta(a)$ but $a I_i b P_i c I_i d$. That is, if the last indifference class is broken into two tiers, there are at least two objects in each "tier".

4. there exist $a, b, c \in H$ such that $a, b, c \in \eta(a)$ but $a P_i b P_i c$. That is, if the last indifference class is broken, there are at least three tiers.

5. there exist a, b such that $a, b \in \eta(a)$ but $a P_i b$ and $b P_j a$. That is, if the last indifference class is broken into two tiers, both strict preferences are present.

Suppose the first case. Consider two agents, 1 and 2. Let all $i \neq 1,2$ top-rank $\eta(w_i)$ and have $i \succ_i \cdots$, so that they retain own endowment. Let R_1 be the new preference. Let $a = w_1$ and $b = w_2$, so that $w_1 I_1 w_2$. Let R_2 be such that 2 top-ranks $\eta(w_1)$. Let $1 \succ_1 2 \succ_1 \cdots$. Finally, Then 1 and 2 do not trade. If 1 instead submits $w_2 P_1' w_1$, then $x_1' = w_2, x_2' = w_1$. This is a strict improvement for 2 without harming 1; thus $Q = \{1,2\}$ forms a coalition.

Suppose the second case. Consider three agents, 1, 2, and 3. Again, let all $i \neq 1,2,3$ top-rank $\eta(w_i)$ and have $i \succ_i \cdots$, so that they retain own endowment. Let w_1, w_2, w_3 be such that the new preference allows $\eta(w_1) = \eta(w_2)$ but allows $w_1 P_i w_2$. Let 1 and 2 top-rank $\eta(w_3)$. Let 3 top-rank $\eta(w_1)$, and $2 \succ_3 1 \succ_3 3 \succ_3 \cdots$. Then 3 and 2 trade, and 1 keeps his own endowment. However, if 3 instead submits the new preference, $w_1 P_3' w_2$ and $w_1 P_3' w_3$, then 3 and 1 trade instead. This benefits $Q = \{1,3\}$.

Suppose the third case. Consider four agents, 1, 2, 3, and 4. Again, let all $i \neq 1,2,3,4$ top-rank $\eta(w_i)$ and have $i \succ_i \cdots$, so that they retain own endowment. Let w_1, w_2, w_3, w_4 be such that under the new preference, $w_1 I_i w_2 P_i w_3 I_i w_4$, and let R_3, R_4 have this preference. Let R_1, R_2 top-rank $\eta(w_1)$. Let $\succ$ be as follows:

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
2	3	1	1
4	2	⋮	⋮
⋮	⋮		

Then $x_1 = w_2, x_2 = w_3, x_3 = w_1, x_4 = w_4$. Now let 2 also submit $w_1 I_2' w_2 I_2' w_3 I_2' w_4$. Then $x_1' = w_4, x_2' = w_2, x_4' = w_1$. This strictly benefits 4 without harming 2, so $Q = \{2,4\}$ is a profitable coalition.

Suppose the fourth case. Consider three agents, 1, 2, and 3. Again, let all $i \neq 1,2,3$ top-rank $\eta(w_i)$ and have $i \succ_i \cdots$, so that they retain own endowment. Let w_1, w_2, w_3 be such that $w_1, w_2, w_3 \in \eta(w_1)$ but the new preference is $w_1 P_i w_2 P_i w_3$. Let R_1, R_2 be indifferent between these objects, and let R_3 be the new preference. Let $\succ$ be as follows:

$\succ_1$	$\succ_2$	$\succ_3$
2	1	⋮
1	3	
3	2	
⋮	⋮	

Then $x_1 = w_2, w_2 = x_1, w_3 = w_3$. Now let 1 submit the new preference, $w_1 P_1' w_2 P_1' w_3$. Then $x_1' = w_1, x_2' = w_3, x_3' = w_2$, strictly benefiting 3 without harming 1, so $Q = \{1,3\}$ is a profitable coalition.

Finally, suppose the fifth case. This is the symmetric-maximality case. Consider two agents, 1 and 2. Let w_1 and w_2 be the relevant objects. Let $R_1 \in \mathcal{R}(\mathcal{H})$, so $w_1 I_1 w_2$; and $w_1 P_2 w_2$. Again, let all $i \neq 1,2$ top-rank $\eta(w_i)$ and have $i \succ_i \cdots$, so that they retain own endowment. Then $x_1 = w_1, x_2 = w_2$. However, if 1 instead submits $w_2 P_1' w_1$, then $x_1' = w_2$ and $x_2' = w_1$. This strictly benefits 2 without harming 1, so $Q = \{1,2\}$ is a profitable coalition.

Now suppose case II is true, but not case I. Then there are $R_*, R_{**} \in \tilde{\mathcal{R}}_i$ which are not compatible with the same $\bar{\mathcal{R}}_i(\mathcal{H})$.

If $R_* \in \bar{\mathcal{R}}_i(\mathcal{H})$ and $R_{**} \in \bar{\mathcal{R}}_i(\mathcal{H}')$ for some $\mathcal{H} \neq \mathcal{H}'$, then there must be some $h, h' \in H$ such that $h I_* h'$ but $h P_{**} h'$, or vice versa. Then we can apply cases I.1 or I.2 from above.

Now suppose $R_* \in \bar{\mathcal{R}}_i(\mathcal{H}, *, \text{int})$ and $R_{**} \in \bar{\mathcal{R}}_i(\mathcal{H}, *', \text{int}')$. That is, they are different commodified goods-plus domains on the same partition. It must be that R_* and R_{**} bottom-rank the same partition element of $\mathcal{H}$ (denoted H_k), else they are compatible in the same $\bar{\mathcal{R}}_i(\mathcal{H})$. Further, it must be that R_* and R_{**} are differing "alternative" rankings over H_k. Therefore there must be $h_1, h_2 \in H_k$ such that $h_1 P_* h_2$ but $h_1 P_{**} h_2$.

Let $w_i = h_i$ for all i; let $R_2 = R_*$; and let R_i top-rank $\eta(w_i)$ for all $i \neq 2$. Finally, let $i \succ_i \cdots$ for all i. Then $\text{TTC}_{\succ}(R) = w$. However, if 1 instead reports $R_1' = R_{**}$, then $x_1' = h_2$ and $x_2' = h_1$, a strict improvement for 2 without harming 1. □