House-Swapping with Commodified Objects*

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Abstract

We study the exchange of indivisible objects ("house-swapping") when the goods may be commodified. In many house-swapping markets, some objects may effectively be indistinguishable from one another, as with dorm rooms or school seats. Thus, all agents are indifferent between copies of the same variety. We call this setting "commodified objects". Top trading cycles (TTC) with fixed tie-breaking has been suggested and used in practice to deal with indifferences in house-swapping problems. However, with general indifferences, TTC with fixed tie-breaking is not Pareto efficient or group strategy-proof. Further, it may not select the strict core, even when it exists. In our setting, agents are *always and only* indifferent between copies of objects. In this setting, TTC with fixed tie-breaking maintains Pareto efficiency, group strategy-proofness, and strict core selection.

1 Introduction

Important markets such as living donor organ transplants, dorm assignments, and school choice can be modeled as "house-swapping" problems. In a house-swapping problem, each agent is endowed with an indivisible object (called a "house") and has preferences over the set of objects. The objective is to sensibly re-allocate these objects among the agents. Monetary transfers are disallowed, and participants have property rights to their own endowments. Shapley and Scarf (1974) first introduce house-swapping when agents have strict preferences over houses. The usual stability notion is the core; an allocation is in the core if no subset of agents would prefer to trade their endowments among themselves. Gale's *top trading cycles* (TTC) algorithm finds an allocation in strong core. Roth and Postlewaite (1977) further show that the strict core is non-empty, unique, and Pareto efficient. Roth (1982) shows that TTC is strategy-proof; Moulin (1995) shows it is group strategy-proof. These properties make TTC a normatively attractive algorithm.

The assumption that preferences are strict is quite strong. In particular, if the houses are not unique, agents should naturally be indifferent. We present a model of house-swapping where there are indistinguishable copies of objects ("types"). The model restricts agents to be always indifferent between copies of the same object, but never between distinct objects. We call this problem "house-swapping with commodified objects". This models important situations where the house-swapping model is applied in practice. For example, in dorm or public housing assignments, many units are effectively the same (same floor plan in the

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same building, for example). Likewise in school assignments, different slots at the same school are indistinguishable. We see commodified goods as a minimalist model of indifferences, where indifferences are most plausible (or perhaps undeniable!).

In the fully general setting where agents' preferences may contain any indifferences, *TTC with fixed tie breaking* is often used in practice; ties in preference orders are broken by some external rule. Abdulkadrioglu and Sönmez (2003) propose it in the setting of school choice with priorities. However, it is not Pareto efficient or group strategy-proof. Indeed, Ehlers (2002) shows that these two properties are not compatible in house-swapping with indifferences. Additionally, the strict core may be empty or non-unique, and TTC with fixed tie breaking may not select a strict core allocation when one exists.

Commodified goods adds structure to the general case of indifferences by constraining any indifferences to be universal among agents and by limiting the set of preference rankings agents may submit. While the strong core still may not exist, it is essentially unique when it does exist. We show that in house-swapping with commodified goods, TTC with fixed tie-breaking recovers Pareto efficiency and group strategy-proofness. It also selects the unique strong core when it exists, and selects an element in the weak core otherwise. We also show that the commodified goods setting is a maximal setting such that these properties hold, in the sense that allowing a superset of possible preference orderings breaks each property.

In summary, we present a reasonable model of indifferences, commodified goods, which can capture settings where house-swapping is used in practice. Further, there is an advantage in working in the commodified goods setting over the more general setting of full indifferences, as TTC with fixed tie breaking preserves Pareto efficiency, core selection, and group strategyproofness. Thus we also provide a strong normative argument for using TTC with fixed tie-breaking in settings like school choice and dorm assignment.

In addition to the papers already mentioned, our paper contributes to a broader literature on object assignment problems. A number of important papers deal with the object allocation problem without endowments; e.g. Pápai (2000) and Ehlers et al. (2002). Recently, others have proposed mechanisms for the house swapping model with indifferences; in particular, Quint and Wako (2004) and Jaramillo and Manjunath (2012).

Section 2 presents the formal notation. Section 3 provides the main results. Section 4 concludes.

2 Model

We present the model primitives. First we recount the classical Shapley and Scarf (1974) domain. Afterwards we introduce our "commodified objects" domain.

We now present the general house-swapping model (with distinguishable objects). Let $N = \{1, \dots, n\}$ be a finite set of agents, with generic member i. Let $H = \{h_1, \dots, h_n\}$ be a set of houses, with generic member h. Every agent is endowed with one object, given by a bijection $w : N \to H$. The set of all endowments is $W(N, H)$ or W for short. An allocation is an assignment of an object to each agent, given by a bijection $x : N \to H$. The set of all allocations is likewise $X(N, H)$ or X. We denote $x(i) = x_i$ and $w(i) = w_i$ for short.

Each agent has preferences R_i over H. A preference profile is $R = (R_1, R_2, ..., R_n)$. Let $\mathcal{R}_i$ be the set of i's possible preferences. If every $\mathcal{R}_i$ is the set of strict preference orderings, $\mathcal{R}_i^N$ is the classical **strict preferences domain**. If every $\mathcal{R}_i$ is the set of weak preference orderings, $\mathcal{R}_i^N$ is the classical **general indifferences domain**.

Our main domain is **commodified objects**. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ be a partition of H. Each agent i has a strict linear order $\geq_i$ over $\mathcal{H}$, and preferences over H are derived from this. Formally, for $h \in H_h$ and $h' \in H_{h'}$,

$$hR_ih' \iff H_h \geq_i H_{h'}$$

The partition $\mathcal{H}$ defines the house types. $\mathcal{R}_i(\mathcal{H})$ is set of preferences given by the partition; we sometimes suppress $(\mathcal{H})$ from the notation when context makes it clear. Given $\mathcal{H}$, $\mathcal{R}(\mathcal{H}) = \mathcal{R}_i(\mathcal{H})^N$ is a commodified objects domain. Note that all agents are indifferent between houses in the same partition and have strict rankings between houses in different partitions. Because of this, we refer to indifference classes for the domain with the understanding that everyone shares the same indifference classes. In this notation, we treat the commodified objects as having identities. I.e. we keep track of the objects in $\mathcal{H}_1$; however, the objects are indistinct and *always* have the same welfare implications.

Commodified objects models settings where some objects are indistinguishable to all participants. An example is dorm rooms or public housing, where there may be many units of the same basic layout and amenities.

2.1 Rules

This subsection recounts formalities on rules (mechanisms) and top trading cycles. Familiar readers may safely skip this subsection.

A market is a tuple (N, H, w, R). A **rule** is a function $f : \mathcal{R} \to X$; given a preference profile, it produces an allocation. When it is unimportant or clear, we suppress inputs from the notation. Denote $f_i(R)$ to be i's allocation; and $f_Q = \{f_i : i \in Q\}$. Fix a rule f and setting. We work with the following desiderata ("axioms").

A rule is Pareto efficient if it always produces Pareto efficient allocations.

Pareto efficiency (PE). For all $R \in \mathcal{R}$, there is no other allocation $x \in X$ such that $x_iR_if_i$ for all $i \in N$ and $x_iP_if_i$ for at least one i.

Strategy-proofness ensures no agent can improve his outcome by submitting false preferences. That is, agents are weakly incentivized to tell the truth.

Strategy-proofness (SP). For all $R \in \mathcal{R}$, for any $q \in N$ and R'_q, $f_q(R)R_qf_q(R'_q, R_{-q})$.

Group strategy-proofness is stronger than SP. It requires that no coalition of agents can improve their outcomes by submitting false preferences. Note that in the following, we require both the true preferences and potential misreported preferences to come from the same set $\mathcal{R}$.

Group strategy-proofness (GSP). For all $R \in \mathcal{R}$, there do not exist $Q \subseteq N$ and R'_Q such that $(R'_Q, R_{-Q}) \in \mathcal{R}$ and $f_q(R'_Q, R_{-Q})R_qf_q(R)$ for all $q \in Q$ with $f_q(R'_Q, R_{-Q})P_qf_q(R)$ for at least one.

Individual rationality models the constraint of voluntary participation. It requires that agents do at least as well as their own endowments.

Individual rationality (IR). For all w and $R \in \mathcal{R}$, $f_iR_iw_i$.

We also define the core, which is a property of allocations. An allocation is in the core if there is no subset of agents who would rather trade their endowments among themselves.

Definition 1. An allocation x is blocked if there exists a coalition $N' \subseteq N$ and allocation x' such that $w(N') = x'(N')$ and for all $i \in N'$, $x'_i R_i x_i$, with $x'_i P_i x_i$ for at least one. An allocation x is in the **core** if it is not blocked.

The weak core is requires that all members of the coalition are strictly better off.

Definition 2. An allocation x is weakly blocked if there exists a coalition $N' \subseteq N$ and allocation x' such that $w(N') = x'(N')$ and for all $i \in N'$, $x'_i P_i x_i$. An allocation x is in the **weak core** if it is not weakly blocked.

The core property models the restriction imposed by property rights. Notice that individual rationality excludes blocking coalitions of size 1. The last axiom is core-selecting.

Core selecting (CS). For all $R \in \mathcal{R}$ and $w \in W$, $f(R)$ is in the core, if the core is nonempty.

We will present results that commodified objects is a largest domain on which TTC$_\succ$ is PE or CS. It is also "essentially" a largest domain on which TTC$_\succ$ is GSP (we will note the technicalities when we present the result). By the "largest domain", we mean the following.

Definition 3. A domain $\mathcal{R}_i^N$ is **(symmetric-) maximal** for an axiom A and a rule f if:

1. f is A on $\mathcal{R}_i^N$
2. for any $\tilde{\mathcal{R}}_i \supsetneq \mathcal{R}_i$, f is not A on $\tilde{R}_i^N$
3. *(For symmetric maximal: if $hP_ih' \notin \tilde{\mathcal{R}}_i$ but $hP_ih' \in \tilde{\mathcal{R}}_i$ then also $h'P_ih \in \tilde{\mathcal{R}}_i$.)*

Note that this definition of maximality depends on both the domain and the rule f, which differs from elsewhere in the literature. Also note that we restrict to the same set of possible preferences for each agent in both the maximal domain and for any expanded domain.

Symmetric maximality requires that if any indifference is broken, then both strict relations are added to the domain. Of course, symmetric maximality is a weaker condition, as it restricts the possible domain expansions. However, it is natural in a preference solicitation mechanism to allow either ranking. The third restriction rules out situations where some agents may have a *strict ranking only in a particular direction*, while others are indifferent.[1] Our results for PE and CS will be with respect to maximality. Our GSP result is with respect to symmetric maximality.

3 Top trading cycles with fixed tie breaking

In this paper, we analyze top trading cycles (with tie breaking) in the settings defined in the previous section. For an extensive history, we refer the reader to Morill and Roth (2024). We briefly define TTC and TTC with fixed tie-breaking.

Algorithm 1. ***Top Trading Cycles****. Consider a market (N, H, w, R) under strict preferences. Draw a graph with N as nodes.*

1. *Draw an arrow from each agent i to the owner (endowee) of his favorite remaining object.*

[1] For example, a seat at a school with a scholarship is strictly preferred to a seat to the same school without one, unless a student has outside ("last-dollar") funds.

2. *There must exist at least one cycle; select one of them. For each agent in this cycle, give him the object owned by the agent he is pointing at. Remove these agents from the graph.*

3. *If there are remaining agents, repeat from step 1.*

We denote this as $TTC(R)$.

TTC is only well defined with strict preferences, as step 1 requires a unique favorite object. In practice, a **fixed tie breaking** rule is often used to resolve indifferences. Given N, let $\succ = (\succ_1, \ldots, \succ_N)$, where each $\succ_i$ is a strict linear order over N. This linear order will be used to break indifferences between objects (based on their owners). Then let $R_{i,\succ}$ be given by the following. For any $j \neq j'$, let $w_j P_{i,\succ} w_{j'}$ if either

1. $w_j P_i w_{j'}$, or

2. $w_j I_i w_{j'}$ and $j \succ_i j'$

Then $R_{i,\succ}$ is a strict linear order over the individual houses. Example 1 illustrates a tie-break rule. Let $R_\succ = (R_{1,\succ}, \ldots, R_{N,\succ})$. Given a fixed tie breaking rule, **TTC with fixed tie breaking (TTC$_\succ$)** is $TTC_\succ(R) = TTC(R_\succ)$. That is, the tie breaking rule is used to generate strict preferences, and TTC is applied to the resulting profile. Formally, each tie breaking profile $\succ$ generates a different TTC with fixed tie breaking rule.

Example 1. Let $N = \{1, 2, 3, 4\}$.

R_1		$\succ_1$		$R_{1,\succ}$
w_3, w_4	$+$	1	$\rightarrow$	w_3
w_1, w_2		2		w_4
		3		w_1
		4		w_2

TTC$_\succ$ is not Pareto efficient in general indifferences. Example 2 gives the simplest case.

Example 2. Let $N = \{1, 2\}$ and preferences be given by $w_1 I_1 w_2, w_1 P_2 w_2$. Let $\succ_i = (1, 2)$ for both agents. In the first round of TTC$_\succ$, both agents point to themselves. The allocation is $x = (w_1, w_2)$, which is Pareto dominated by $x' = (w_2, w_1)$.

The example illustrates the problem with indifferences – TTC$_\succ$ may not take advantage of Pareto gains made possible by the indifferences. The commodified objects domain rules out these situations. Our first main result is that TTC$_\succ$ is Pareto efficient in commodified objects.

Proposition 1. *$TTC_\succ$ is PE for all tie-breaking profiles $\succ$ in any $\mathcal{R}(\mathcal{H})$. Further, each $\mathcal{R}(\mathcal{H})$ is a maximal domain on which $TTC_\succ$ is PE for all $\succ$.*

Proof. Appendix. □

The intuition is that commodified goods rules out situations like in Example 2. In contrast, under commodified goods, agent 1 will have to leave his allocated indifference class in order to benefit agent 2. Any more general domain will reintroduce this possibility.

Our second result deals with the core. In general indifferences, the set of core allocations may not be a singleton. There may be no core allocations or multiple. Likewise, the set of core allocations may be empty or multi-valued in commodified objects, as Example 3 illustrates. However, the core is *essentially unique* when it exists, in that all core allocations are re-arrangements of indistinguishable copies.

Example 3. Let R be given by the following.

R_1	R_2	R_3
w_2, w_3	w_1	w_1
w_1	w_2, w_3	w_2, w_3

It is straight forward to check that the core is empty.

Furthermore, $\text{TTC}_\succ$ always selects the core for any tie-breaking rule $\succ$ in commodified goods. This is in contrast to the result from Ehlers (2014) for general indifferences, where only the weak core is guaranteed.

Proposition 2. *Let $x = TTC_\succ(R)$ and $R \in \mathcal{R}(\mathcal{H})$. For any $\succ$,*

1. *x is in the weak core.*
2. *if the core exists, then x is in the core. That is, $TTC_\succ$ is CS.*
3. *if y is in the core, then $x_i I_i y_i$ for all $i \in N$.*

Further, each $\mathcal{R}(\mathcal{H})$ is a maximal domain on which $TTC_\succ$ is CS.

Point 3 is the "essential uniqueness" of the core. Since indifferences are universal, it says that all core assignments are rearrangements of copies of object types. Point 2 is implied by 3, but listed separately for clarity. In summary, the $\text{TTC}_\succ$ always produces an allocation in the weak core, produces an allocation in the core when it exists, and the core allocation is unique up to the identities of the commodified objects.

Proof. Appendix. □

The intuition is to that for Pareto efficiency; note any allocation in the core is Pareto efficient. Under general indifferences, the core may be multi-valued due to re-arranging objects that agents are indifferent between. Under commodified goods, this is simply re-arranging copies of indistinguishable objects.

Our third result is that $\text{TTC}_\succ$ is group strategyproof in commodified objects. $\text{TTC}_\succ$ is not GSP in general indifferences. Example 4 illustrates; an agent can break his own indifference to benefit a coalition member without harming himself.

Example 4. Let R and R' be given by the following, and let $Q = \{1, 3\}$.

R_1	R_2	R_3	R'_1
w_2, w_3	w_1	w_1	w_3
w_1	w_2	w_2	w_1, w_2
	w_3	w_3	

Let $\succ_i = (1, 2, 3)$ for all i. Then $\text{TTC}_\succ(R) = (w_2, w_1, w_3)$. But if 1 misreports R'_1, then $\text{TTC}_\succ(R') = (w_3, w_2, w_1)$. Then 1 is indifferent, and 3 is strictly better off.

Commodified objects eliminates possibilities like the above in a subtle way. The model imposes "exogenous" indifferences; agents can *only* report they are indifferent between all objects in the same indifference class given by $\mathcal{H}$.[2]

Proposition 3. *$TTC_{\succ}$ is GSP for all tie-breaking profiles $\succ$ in any $\mathcal{R}(\mathcal{H})$. Further, each $\mathcal{R}(\mathcal{H})$ is a symmetric-maximal domain on which $TTC_{\succ}$ is GSP.*

Proof. Appendix. □

The proof is similar to that in Moulin (1995), relying on the restriction that agents can only report their indifference class and not arbitrarily break ties within them. This rules out cases like Example 4. A profitably deviating coalition would require a "first mover" misreport in order to obtain a welfare equivalent but distinct object (see also the proof in Bird, 1984; Sandholtz and Tai, 2024), which commodified goods rules out. We also note that $\mathcal{R}(\mathcal{H})$ is *not* a maximal domain on which $\text{TTC}_{\succ}$ is GSP with the following example.

Example 5. Consider $H = \{h_1, h_2\}$ and $\mathcal{H} = \{\{h_1, h_2\}\}$. Let $\mathcal{R}' = \mathcal{R}(\mathcal{H}) \cup (h_1 P h_2)$. That is, expand the domain by including the ordering $h_1 P h_2$. It can be verified that $\text{TTC}_{\succ}$ is still group strategyproof.

If $R_1 = R_2 = (1P1)$ or $R_1 = R_2 = (1I2)$, then of course there is no possible group manipulation. Now let $h_1 = w_1$ and $h_2 = w_2$, and $1 \succ_i 2$ for both i. Consider two possible (true) preference profiles:

R_1	R_2		R_1	R_2
h_1	h_1, h_2	or	h_1, h_2	h_1
h_2				h_2

In the first case, there is no improving allocation. In the second case, it would be advantageous for agent 1 to claim h_2 and pass along h_1, but this is not possible, since this preference is not in $\mathcal{R}'$. Now let $h_1 = w_2$ and $h_2 = w_1$. In the first case, there is again no improving allocation (they trade in $\text{TTC}_{\succ}$). In the second case, there is again no improving allocation.

The following theorem collects the results presented above.

Theorem 1. *$TTC_{\succ}$ is PE, CS, and GSP for all tie-breaking profiles $\succ$ in any $\mathcal{R}(\mathcal{H})$. Each $\mathcal{R}(\mathcal{H})$ is a maximal domain on which $TTC_{\succ}$ is PE and CS. Each $\mathcal{R}(\mathcal{H})$ is a symmetric-maximal domain on which $TTC_{\succ}$ is GSP. Additionally, if the core of (N, w, R) for $R \in \mathcal{R}(\mathcal{H})$, it is unique up to the identity of the objects.*

The results and examples suggest a practical issue – selection of $\mathcal{H}$ given the set of objects H. In some cases the commodification may be obvious; e.g., identical tasks or slots in a program. In other cases, there may be some ambiguity; e.g., are two dorms of the same floor plan but on different floors equivalent? Inappropriately combining two indifference classes can lead to efficiency losses in the spirit of Example 2. On the other hand, splitting an indifference class can allow group manipulations like in Example 4. We leave formal results on the tradeoff as future work.

[2]Constraining the reports is also an important difference from Ehlers (2002).

4 Conclusion

The house-swapping market is a classic model in economic theory with applications to important markets like housing assignment, school choice, and organ exchange. Surprisingly, it took about thirty years from Shapley and Scarf (1974) to generalize results to indifferences. Since then, there has been a significant amount of work dealing with indifferences.

TTC with fixed tie-breaking is a commonly used mechanism for house-swapping problems with indifferences. Unfortunately, it does not preserve Pareto efficiency, group strategyproofness, or core selection in general indifferences.

We have proposed a model of a particular kind of indifferences, "commodified objects", where there are indistinguishable copies of objects. Commodified objects captures many of the situations where house-swapping is relevant. (Consider for example housing assignment with many indistinguishable dorm rooms.) Therefore it is a compelling case to include in a model of house-swapping.

Fortunately, TTC with fixed tie-breaking preserves the aforementioned properties – Pareto efficiency, group strategyproofness, and core selection – on commodified objects. Moreso, Pareto efficiency and core selection fail on any larger domains. While group strategyproofness is preserved on some larger domains, it fails on any "symmetrically larger" domain. Thus commodified objects is not only a compelling case to include, but also the most general case preserving these properties.

We leave a characterization of TTC with fixed tie-breaking on commodified goods also remains an open question.

References

Abdulkadrioglu, A., & Sönmez, T. (2003). School choice: A mechanism design approach. *American Economic Review*, *93*(3).

Bird, C. G. (1984). Group incentive compatibility in a market with indivisible goods. *Economics Letters*, *14*.

Ehlers, L. (2002). Coalitional strategy-proof house allocation. *Journal of Economic Theory*, *105*(2), 298–317.

Ehlers, L. (2014). Top trading with fixed tie-breaking in markets with indivisible goods. *Journal of Economic Theory*, *151*, 64–87.

Ehlers, L., Klaus, B., & Pápai, S. (2002). Strategy-proofness and population monotonicity for house allocation problems. *Journal of Mathematical Economics*, *38*.

Jaramillo, P., & Manjunath, V. (2012). The difference indifference makes in strategy-proof allocation of objects. *Journal of Economic Theory*, *147*(5), 1913–1946.

Morill, T., & Roth, A. E. (2024). Top trading cycles. *Journal of Mathematical Economics*, *112*.

Moulin, H. (1995). *Cooperative microeconomics: A game-theoretic introduction*. Prineton University Press.

Pápai, S. (2000). Strategyproof assignment by hierarchical exchange. *Econometrica*, *68*.

Quint, T., & Wako, J. (2004). On Houseswapping, the Strict Core, Segmentation, and Linear Programming. *Mathematics of Operations Research*, *29*(4), 861–877.

Roth, A. E. (1982). Incentive compatibility in a market with indivisible goods. *Economics Letters*, *9*(2), 127–132.

Roth, A. E., & Postlewaite, A. (1977). Weak versus strong domination in a market with indivisible goods. *Journal of Mathematical Economics*, *4*(2), 131–137.

Sandholtz, W., & Tai, A. (2024). Group incentive compatibility in a market with indivisible goods: A comment. *Economics Letters*, *243*.

Shapley, L., & Scarf, H. (1974). On cores and indivisibility. *Journal of Mathematical Economics*, *1*(1), 23–37.

Takamiya, K. (2001). Coalition strategy-proofness and monotonicity in shapley–scarf housing markets. *Mathematical Social Sciences*, *41*.

Appendix

The appendix contains the proofs of the results in the main text. Throughout, for a partition $\mathcal{H}$, let $\eta : H \to \mathcal{H}$ associate an object h with its indifference class under $\mathcal{R}(\mathcal{H})$. Additionally, given a market and $\text{TTC}_{\succ}(R)$, denote $S_k(R)$ as the kth cycle executed in $\text{TTC}_{\succ}(R)$.[3]

It is immediate that $\text{TTC}_{\succ}$ is IR, as any agent pointing at his own endowment must be assigned to it. We will use this fact for the some of the proofs.

Proposition (Proposition 1)**.** *$TTC_{\succ}$ is PE for all tie-breaking profiles $\succ$ in any $\mathcal{R}(\mathcal{H})$. Further, each $\mathcal{R}(\mathcal{H})$ is a maximal domain on which $TTC_{\succ}$ is PE for all $\succ$.*

Proof. The result is trivial for $|N| = 1$. Now let $|N| \geq 2$.

We show that PE is satisfied on commodified goods. Consider any (N, H, w). Let $\mathcal{H}$ be any partition and let $R \in \mathcal{R}(\mathcal{H})$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose it the partition has at least two subsets. Let $x = TTC_{\succ}(R)$, and suppose $y \in X$ Pareto dominates x. Let $W = \{i : y_i P_i x_i\}$ be the set of agents who strictly improve under y, which must be nonempty. Let $i \in W$ be the first agent in W assigned in $TTC_{\succ}(R)$.

Denote $i \in S_k(R)$ and $\eta(y_i) = H_y$. We have that $y_i P_i x_i$. . Note that at step k, no objects in H_y were available, otherwise i would have pointed to one of them rather than at x_i.

Since $y_i P_i x_i$, we have $x_i \notin H_y$ but $y_i \in H_y$. Thus there must be another agent j such that $x_j \in H_y$ but $y_j \notin H_y$. Since y Pareto dominates x, $y_j R_j x_j$. Since y_j and x_j are not in the same indifference class, we have $y_j P_j x_j$. (This is where commodified objects is used.)

Then $j \in W$. Further, j must have been assigned before step k, since no object in H_y was available at step k. This contradicts the presumption that i was the first agent in W assigned.

We now prove that any $\mathcal{R}(\mathcal{H})$ (just denoted $\mathcal{R}$ here) is maximal on which $\text{TTC}_{\succ}$ is PE. Let $\tilde{\mathcal{R}}_i \supset \mathcal{R}_i$ and denote $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_i^N$. We will show there exists a w, $R \in \tilde{\mathcal{R}}$, and $\succ$ such that $x = \text{TTC}_{\succ}(R)$ is not PE.

It must be that $\tilde{\mathcal{R}}_i$ contains some preference orderings R', R'' such that for $h_1, h_2 \in H$ we have $h_1 I' h_2$ but $h_1 P'' h_2$. Thus $\tilde{\mathcal{R}}$ contains a preference profile such that $h_1 I_1 h_2$ but $h_1 P_2 h_2$. (The labels 1 and 2 are without loss of generality.)

Let $w_1 = h_1$ and $w_2 = h_2$. Let R_j for $j \in \{3, \ldots, N\}$ top-rank w_j. Finally, let $\succ_i$ have i as the highest priority for all i. It is straight-forward that $\text{TTC}_{\succ}(R) = w$. However, this is Pareto dominated by the allocation where 1 and 2 trade assignments. □

Proposition (Proposition 2)**.** *Let $x = TTC_{\succ}(R)$ and $R \in \mathcal{R}(\mathcal{H})$. For any $\succ$,*

1. *x is in the weak core.*
2. *if the core exists, then x is in the core. That is, $TTC_{\succ}$ is CS.*
3. *if y is in the core, then $x_i I_i y_i$ for all $i \in N$.*

Further, each $\mathcal{R}(\mathcal{H})$ is a maximal domain on which $TTC_{\succ}$ is CS.

Proof. We first note a fact about $x = \text{TTC}_{\succ}(R)$. If $i \in S_\ell(R)$ and $h P_i x_i$, then h was assigned in a cycle before ℓ. This follows from the definitions; $h P_i x_i$ implies $h P_{i,\succ} x_i$, and $\text{TTC}_{\succ}(R) = \text{TTC}(R_{\succ})$. Under $\text{TTC}(R_{\succ})$, an object $h P_{i,\succ} x_i$ must have been assigned earlier than ℓ, otherwise i would have pointed to it.

[3] Note that S_k may not be unique, since multiple cycles may appear in step 2 of Algorithm 1.

(1.) Let $R \in \mathcal{R}(\mathcal{H})$ and denote $x = \text{TTC}_{\succ}(R)$. Suppose there is a weakly blocking coalition $N' \subseteq N$ with allocation y such that $y_i P_i x_i$ for all $i \in N$. We show by induction on the cycles of $\text{TTC}_{\succ}(R)$ that N is empty.

Step 1. All $i \in S_1(R)$ received one of their top-ranked objects, so they cannot be in N'.

Step k. Suppose N' does not include any members of earlier cycles. Now consider $i \in S_k(R)$. If $y_i P_i x_i$, then y_i must be an object assigned in $\cup_{\ell=1}^{k-1} S_\ell(R)$. But no agents in $\cup_{\ell=1}^{k-1} S_\ell(R)$ are in N', so it is not feasible to include i in N' either. Thus no agents in $S_k(R)$ are in N'.

Then N' is empty, completing the proof of this claim.

(2.) 2 is implied by 3.

(3.) Suppose the core of (N, H, w, R) is nonempty and contains y. Denote $x = \text{TTC}_{\succ}(R)$. We show $x_i I_i y_i$ $(*)$ for all i by induction on the cycles.

Step 1. All $i \in S_1(R)$ received one of their top-ranked objects, so $x_i R_i y_i$. Suppose $(*)$ is not true for $S_1(R)$. Then there is some $i \in S_1(R)$ such that $x_i P_i y_i$. But then $S_1(R)$ and x block against y, a contradiction.

Step k. Suppose that $(*)$ is true for all cycles before k. Suppose for some $i \in S_k(R)$ we have $y_i P_i x_i$. Then y_i was assigned in a cycle before k. Further, y_i and x_i are in different indifference classes. Thus under y if y_i is assigned to i, an agent j in $(\cap_{\ell=1}^{k-1} S_\ell(R)) \cap H_{y_i}$ must be assigned an object outside of H_{y_i}. But then it cannot be that $y_j I_j x_j$, a contradiction.[4] Thus we have that $x_i R_i y_i$ for all $i \in S_k(R)$. Suppose $(*)$ is not true for $S_k(R)$. Then there is some $i \in S_k(R)$ such that $x_i P_i y_i$. But then $S_k(R)$ and x block against y, a contradiction.

Then $x_i I_i y_i$ for all i, as desired.

We now prove the maximality claim. Towards a contradiction, the setup is the same as in the proof of Proposition 1. We recount it here for convenience. Denote $\mathcal{R}(\mathcal{H}) = \mathcal{R}$. Let $\tilde{\mathcal{R}}_i \supsetneq \mathcal{R}_i$ and denote $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_i^N$. It must be that $\tilde{\mathcal{R}}$ contains a preference profile such that $h_1 I_1 h_2$ but $h_1 P_2 h_2$, and either R_1 or R_2 is in $\mathcal{R}_i$.

Let $w_1 = h_1$ and $w_2 = h_2$. Let R_j for $j \in \{3, \ldots, N\}$ top-rank w_j. Finally, let $\succ_i$ have i as the highest priority for all i. It is straight-forward that $\text{TTC}_{\succ}(R) = w$, which is blocked by $x = (w_2, w_1, w_3, w_4, ..., w_N)$. We seek to show x is in the strict core for some $R \in \tilde{\mathcal{R}}$ compatible with the above.

Let exactly one of R_1, R_2 be in $\mathcal{R}_i$.

If $R_1 \in \mathcal{R}_i$, then $\eta(w_1) = \eta(w_2)$. Further, we can suppose that R_1 top-ranks w_1 and its indifference class (since this preference exists in the original domain). Also suppose that w_1 is the highest ranked member of $\eta(w_1)$ under R_2. This is without loss of generality since 1's identity was generic. In order for 2 to receive a strictly better object, he must receive an object outside of $\eta(w_1) = \eta(w_2)$. Recall that $N \setminus \{1, 2\}$ all received their own endowment, which was a favorite object, and 1 received an object in $\eta(w_1)$ which is likewise a favorite. In any potential blocking coalition, 2 must receive some $h \notin \eta(w_1)$. Thus some other member of this coalition must be endowed with some object in $\eta(h)$ but receive an object not in $\eta(h)$, making him strictly worse off.

Now suppose $R_2 \in \mathcal{R}_i$. Then $\eta(w_1) \neq \eta(w_2)$. Further, suppose that there do not exist h, h' such that $h P_1 h'$ but $\eta(h) = \eta(h')$, else we can apply the previous case. Suppose that 2 top-ranks w_1 and its indifference class, so all $N \setminus \{1\}$ received a favorite object. Again, $N \setminus \{1, 2\}$ all received their own endowment, which

[4] This is where commodified goods is used – this claim fails in general indifferences.

was a favorite object. Then in any potential blocking coalition, 1 must receive an object h strictly better than w_2. So $h \notin \eta(w_2)$. Thus some other member of this coalition must be endowed with an object in $\eta(h)$ but receive an object not in $\eta(h)$, making him strictly worse off. $\square$

We now present the proof that $\text{TTC}_\succ$ is GSP in commodified objects. The proof follows Moulin (1995) (Lemma 3.3), which proves GSP for strict preferences. It is illustrative to see where the restriction on commodified goods is used to preserve the logic.

Proposition (Proposition 3)**.** *$TTC_\succ$ is GSP for all tie-breaking profiles $\succ$ in any $\mathcal{R}(\mathcal{H})$. Further, each $\mathcal{R}(\mathcal{H})$ is a symmetric-maximal domain on which $TTC_\succ$ is GSP.*

Proof. Let $R \in \mathcal{R}(\mathcal{H})$, and denote $x = \text{TTC}_\succ(R)$. Let $S_t(R)$ be the cycles of $\text{TTC}_\succ(R)$. Suppose $Q \subseteq N$ is a coalition reporting $R' \in \mathcal{R}(\mathcal{H})$, and denote $x' = \text{TTC}_\succ(R')$. Suppose that $x' R_i x$ for all $i \in Q$. The proof is by induction on the cycles of $\text{TTC}_\succ(R)$ containing Q.

Let t^1 be the smallest index such that $S_{t^1}(R) \cap Q$ is nonempty; that is, the first cycle where member(s) of Q are assigned. By definition of $\text{TTC}_\succ$, for all $j \in \cup_{\ell=1}^{t^1-1} S_\ell(R)$, $x_j = x'_j$. Thus at step t^1, the same objects remain under both $\text{TTC}_\succ(R)$ and $\text{TTC}_\succ(R')$. For $i \in S_{t^1}(R) \cap Q$, the best remaining objects are those in $\eta(x_i)$. But i already reported $\eta(x_i)$ and received some $x_i \in \eta(x_i)$. Thus $x_i = x'_i$ and at step $t^1 + 1$, the same set of objects remain under both $\text{TTC}_\succ(R)$ and $\text{TTC}_\succ(R')$. (This is where commodified goods is applied. This step fails under general indifferences, as i could report and obtain a welfare equivalent but distinct object.)

Now consider the step t^ℓ such that $S_{t^\ell}(R) \cap Q$ is nonempty. Suppose that $x_i = x'_i$ for all $i \in S_{t^k}$ for $k \in \{1, ..., \ell - 1\}$. Then at step t^ℓ, the same objects remain under both $\text{TTC}_\succ(R)$ and $\text{TTC}_\succ(R')$. The remainder of the argument follows exactly as above.

We now prove the symmetric-maximality claim. Again, the setup is the same as in Proposition 1. Let $\tilde{\mathcal{R}}_i \supset \mathcal{R}_i$ and denote $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_i^N$. We will show there exists a w, $R \in \tilde{\mathcal{R}}$, and $\succ$ such that $x = \text{TTC}_\succ(R)$ is not GSP.

It must be that $\tilde{\mathcal{R}}_i$ contains some preference orderings R', R'' such that for $h_1, h_2 \in H$ we have $h_1 I' h_2$ but $h_1 P'' h_2$. Thus $\tilde{\mathcal{R}}$ contains a preference profile such that $h_1 I_1 h_2$ but $h_1 P_2 h_2$. (The labels 1 and 2 are without loss of generality.)

By the symmetry requirement, we also have $h_2 P'' h_1$. If R'' was already in $\mathcal{R}$, then $h_2 P'' h_1$ was already included. If not, then R'' is new and must add both relations.

Let $w_1 = h_1$ and $w_2 = h_2$. Let R_j for $j \in \{3, \dots, N\}$ top-rank w_j. Finally, let $\succ_i$ have i as the highest priority for all i. It is straight-forward that $\text{TTC}_\succ(R) = w$. Let $Q = \{1, 2\}$, $R'_1 = h_2 P'_1 h_1$, and $R'_2 = R_2$. Then $\text{TTC}_\succ(R') = (h_2, h_1, w_3, ..., w_N)$. Then $x'_1 R_1 x_1$ and $x'_2 P_2 x_2$, so this was a profitable group manipulation. $\square$

A Other results

We introduce (Maskin) monotonicity. Let $L(x_i, R_i) = \{h \in H : x_i R_i h\}$ be the lower contour set of R_i at x_i. The definition is standard; a rule f is monotone if when any set of agents move up their allocations in their rankings, the allocation remains the same.

Monotonicity (MON). Let $x = f(R)$. For all $R \in \mathcal{R}$, if $R' \in \mathcal{R}$ is such that for all $i \in N$ we have $L(x_i, R_i) \subseteq L(x_i, R'_i)$, then $x = f(R')$.

Proposition 4. *$TTC_{\succ}$ is MON for all tie-breaking profiles $\succ$ (for any complete, transitive, and reflexive preferences).*

Proof. Takamiya (2001) shows that TTC satisfies MON on strict preferences. Then $\text{TTC}(R_{\succ})$ is MON *according to* $R_{\succ}$.

Since $\succ$ is an exogenous tie-breaking rule, $L(x_i, R_{i,\succ}) \subseteq L(x_i, R'_{i,\succ})$ if and only if $L(x_i, R_i) \subseteq L(x_i, R'_i)$. That is, if x_i moves up the ranking according to $R'_{i,\succ}$, it must have moved up according to R'_i, and vice versa.

We also have that $\text{TTC}(R_{\succ}) = \text{TTC}(R'_{\succ})$ if and only if $\text{TTC}_{\succ}(R) = \text{TTC}_{\succ}(R')$. This is by definition, as $\text{TTC}(R_{\succ}) = \text{TTC}_{\succ}(R)$ and $\text{TTC}(R'_{\succ}) = \text{TTC}_{\succ}(R')$.

Thus MON according to $R_{\succ}$ implies if $L(x_i, R_i) \subseteq L(x_i, R'_i)$ then $\text{TTC}(R_{\succ}) = \text{TTC}(R'_{\succ})$, as desired. $\square$