Continuity of stable marriage correspondences

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Abstract

We study a continuum version of the stable marriage problem, in which each agent has a variable mass to be paired to the opposite side. An agent's mass is interpretable as the divisible time of a single atomistic agent, or as a mass of infinitesimal agents of a particular type. Our object of interest is the mapping between the vector of masses to the set of stable matches. As in the canonical stable marriage model, the set of stable matches can be multi-valued, so this mapping is a correspondence. This correspondence is continuous if and only if the stable match is unique, implying that small changes in the masses may cause discontinuous jumps in the set of stable matches.

1 Introduction

The stable marriage problem of Gale and Shapley (1962) is the canonical two-sided matching market. In their model, two sides of discrete agents seek to match with each other in a way that is *stable*, in the sense that no man and woman who are not paired would be better off by marrying each other. Baïou and Balinski (2002) extend this to a *stable allocation* problem between "workers" and "employers". Each agent remains discrete, but there is a divisible mass of each agent. The goal is to allocate the time of workers to employers in a stable way, similarly defined – no mass of a pair of agents would break off and re-match with each other. The mass of an agent can be alternatively interpreted as infinitesimal individuals of a given type.

It is a well known informal fact of the Gale and Shapley model that small changes to the market composition (e.g. addition or removal of agents) can result in dramatic changes to the set of stable matches.[1] We extend and formalize this result to the stable allocation setting, taking advantage of the setting's additional topological structure. Like in the discrete case, the set of stable matches

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[1]For a formal result of this flavor, see Ashlagi, Kanoria, and Leshno (2016).

may be multi-valued. Thus the mapping from the vector of type masses ("populations") to the set of stable matches is a set-valued function (**correspondence**). Our main result is that this correspondence is continuous if and only if it is single-valued; that is, when the stable matching is unique. An informal statement is below.

Theorem. *The stable matching correspondence $f(\cdot)$ is continuous (upper and lower hemicontinuous) at a population vector p if and only if $f(p)$ is single-valued.*

Before we present the model and technical results, we note some applications of the setting and our results.

Gig platforms. On gig platforms, workers and consumers are plentiful and can modeled symmetrically. The platform seeks to match the time of workers to the needs of consumers.[2]

Stable allocation of tasks. One side is workers, and the other is employers (or projects). Rather than having a mass of agents, each discrete agent has a mass of available or required time.

Marriage markets. Hiller, Wu, and Zhang (2023) build a similar continuum marriage market with four discrete types per side.[3] They note that small demographic shifts can drastically alter results. Our main theorem isolates a source of change; since the stable matching correspondence is not generally continuous, even infinitesimal changes to the population vector can cause large shifts in the set of stable matches.

Negative implications. In the model we study, $f(p)$ may be multi-valued. This introduces two sources of difficulties for comparative statics: selection of a stable match inside $f(p)$ matters; and changes to p can cause discontinuous changes in $f(p)$.

Positive implications. There are special cases where $f(p)$ is single-valued. Baïou and Balinski (2002) note a sufficient condition for a unique stable matching, called "strong nondegeneracy". Informally, a market is strongly nondegenerate if neither it nor any proper subset has equal mass of the two sides. If the populations are drawn from real numbers (rather than integers), uniqueness and thus continuity is the typical case.

Another immediate case is where at least one side shares identical preferences; e.g. all the men types have the same preferences over the women. In these cases, $f(p)$ is continuous.

A number of other papers deal with two sided matching in continuum settings, including Carmona and Laohakunakorn (2024), Che, Kim, and Kojima (2019), Greinecker and Kah (2021), and Menzel (2015).

Section 2 formally describes the model. Section 3 describes useful structure of the set of stable matches. Section 4 contains the main theorem. Section 5 describes an implication of our main result to the approximation of large discrete markets with continuum markets.

[2] This contrasts with the model in Azevedo and Leshno (2016), where one side is plentiful (students), and the other is finite (schools).

[3] Their model neither contains nor is contained by ours; we refer readers to their paper for details.

2 Model

We study a two-sided marriage market with discrete types, which determine preferences, and masses of each type. The model is similar to Baïou and Balinski (2002).[4]

There is a finite set M of man types, each type with a strict preference order $\succ_m$ over $W \cup \{\phi\}$. Likewise, a finite set W of women types, each with a strict preference order $\succ_w$ over $M \cup \{\phi\}$. We use ϕ to represent being unmatched, and suppress the set brackets in the future. There is a vector of positive masses of each type (**population**) $p \in \mathcal{P} \subset \mathbb{R}^{M \cup W}_{++}$, where $\mathcal{P}$ is compact.

An **aggregate matching** is a matrix $Q \in \mathbb{R}^{(M\cup\phi)\times(W\cup\phi)}_{+}$, where $Q_{m,w}$ represents the mass of type m matched to type w. $Q_{\phi,\phi} = 0$ by convention. An aggregate matching is **feasible** if $\sum_{w \in W\cup\phi} Q_{m,w} = p_m$ for all $m \in M$ and $\sum_{m\in M\cup\phi} Q_{m,w} = p_w$ for all $w \in W$. Given Q, we denote the support of Q as $\text{supp}(Q) := \{(m,w) : Q_{m,w} > 0\}$.

Definition 1. A pair (m, w) **blocks** Q if there exist $(m', w') \in (M \cup \phi) \times (W \cup \phi)$ such that

$$Q_{m,w'} > 0, \quad Q_{m',w} > 0$$

and

$$w \succ_m w', \quad m \succ_w m'.$$

Q is **pairwise stable (PS)** if there does not exist any blocking pair. Q is **individually rational (IR)** if, whenever $Q_{m,w} > 0$ for $m \in M, w \in W$, then $w \succeq_m \phi$ and $m \succeq_w \phi$. Q is **stable** if it is PS and IR.

We are interested in the stable matching correspondence $f : \mathcal{P} \rightrightarrows \mathcal{Q}$ defined as

$$f(p) = \{Q : Q \text{ is stable and feasible at } p\}.$$

We recount the definitions of hemicontinuity.

Definition 2. Let f have closed values and compact codomain. Then f is **upper hemicontinuous (uhc)** at $p \in \mathcal{P}$ if and only if for every sequence $p^n \to p$ in $\mathcal{P}$ and every sequence $Q^n \in f(p^n)$ such that $Q^n \to Q$, we have $Q \in f(p)$.

Definition 3. The correspondence f is **lower hemicontinuous (lhc)** at $p \in \mathcal{P}$ if and only if for every sequence $p^n \to p$ in $\mathcal{P}$ and every $Q \in f(p)$, there exists a subsequence p^m and sequence $Q^m \in f(p^m)$ such that $Q^m \to Q$.

Remark 1. These are not the typical primitive definitions of uhc and lhc, but they are equivalent. We will show that $f(p)$ is closed.

[4] The main difference is notational – the unmatched option ϕ is not explicitly encoded; agents are instead allowed to be undermatched relative to their mass.

3 Structure of $f(p)$

Many results on the structure of stable matches are analogous to standard results for the original Gale and Shapley setting; we refer the reader to Roth and Sotomayor (1992).

Using first-order stochastic dominance, we extend the preference $\succeq_i$ of each type i to a partial order $\geq_i$ over matchings as follows. Let $Q^1, Q^2 \in f(p)$ be two matchings.

- $\forall m \in M$: Let $Q^1_m \geq_m Q^2_m$ if for all $\bar{w} \in W \cup \{\emptyset\}$:

$$\sum_{w \in W \cup \{\emptyset\}: w \succeq_m \bar{w}} Q^1_{m,w} \geq \sum_{w \in W \cup \{\emptyset\}: w \succeq_m \bar{w}} Q^2_{m,w}$$

- $\forall w \in W$: Let $Q^1_w \geq_w Q^2_w$ if for all $\bar{m} \in M \cup \{\emptyset\}$:

$$\sum_{m \in M \cup \{\emptyset\}: m \succeq_w \bar{m}} Q^1_{m,w} \geq \sum_{m \in M \cup \{\emptyset\}: m \succeq_w \bar{m}} Q^2_{m,w}$$

Now we define a partial order for each side over matchings.

- Let $Q^1 \geq_M Q^2$ if $\forall m \in M : Q^1_m \geq_m Q^2_m$
- Let $Q^1 \geq_W Q^2$ if $\forall w \in W : Q^1_w \geq_w Q^2_w$

The following are due to Baïou and Balinski (2002).

Lemma 1 (Existence. Baïou and Balinski, 2002. Corollary 1.)**.** *For any p, $f(p)$ is nonempty.*

Lemma 2 (Opposing interests. Baïou and Balinski, 2002. Theorems 3, 4, and 6.)**.** *Let $Q^1, Q^2 \in f(p)$. Then $Q^1 \geq_M Q^2$ if and only if $Q^1 \leq_W Q^2$. Further, there exists a unique $\geq_M$-maximal matching, denoted Q^M, and a unique $\geq_W$-maximal matching Q^W.*

Lemma 3 (Rural hospitals. Baïou and Balinski, 2002. Lemmas 8-9.)**.** *In any $Q^1, Q^2 \in f(p)$,*

$$\begin{aligned} Q^1_{m,\phi} &= Q^2_{m,\phi} && \forall m \\ Q^1_{\phi,w} &= Q^2_{\phi,w} && \forall w. \end{aligned}$$

Further, if m has $Q_{m,\phi} > 0$ in some $Q \in f(p)$, then $Q_{m,w} = Q'_{m,w}$ for all $Q' \in f(p)$. The analogous is true for any w.

Lemma 4 (Strong nondegeneracy. Baïou and Balinski, 2002. Theorem 9.)**.** *Fix M, W, and p. Suppose for all nonempty subsets $M' \subseteq M$ and $W' \subseteq W$,*

$$\sum_{m \in M'} p_m \neq \sum_{w \in W'} p_w.$$

Then $f(p)$ is single-valued.

We now present original results that are useful for proving the main result. First we introduce an alternative characterization of $f(p)$.

Definition 4. Let a **cutoff profile** be $c = (c_m, c_w)_{M,W}$ where $c_m \in W \cup \phi$ and $c_w \in M \cup \phi$ and

$$\forall (m,w): \quad c_m \succeq_m w \text{ or } c_w \succeq_w m \quad \text{and} \quad c_m \succeq_m \phi, c_w \succeq_w \phi.$$

Given c, let $E(c)$ be the allowed cells; i.e.

$$(m,w) \in E(c) \iff w \succeq_m c_m \text{ and } m \succeq_w c_w$$

including (m, ϕ) if $c_m = \phi$, etc. Given a cutoff profile c and population p, let define

$$P_c(p) := \left\{ Q \geq 0 : \begin{array}{c} \sum_w Q_{m,w} = p_m \\ \sum_m Q_{m,w} = p_w \\ Q_{m,w} = 0 \text{ if } (m,w) \notin E(c) \end{array} \right\}$$

Notice that this is a closed polyhedron. The cutoff is **binding** for Q if $Q_{m,c_m} > 0$ and $Q_{c_w,w} > 0$ for all w, m. Denote $\mathcal{C}$ to be the set of cutoff profiles, which is finite because M and W are finite. Note this is $\mathcal{C}(\succeq)$, but we suppress the dependence.

We show that all matchings in $P_c(p)$ are stable, and thus $f(p)$ can be characterized as the union of $P_c(p)$ across all cutoff profiles given $\succeq$. Observe also that $P_c(p)$ is closed.

Proposition 1. *Given c, every $Q \in P_c(p)$ is stable.*

Proof. Suppose c is admissible and $Q \in P_c(p)$. Suppose (m,w) is a blocking pair; then there exists (m', w') such that

$$Q_{m,w'} > 0, \ Q_{m',w} > 0$$

and

$$w P_m w', \ m P_w m'$$

By definition of $P_c(p)$, $w' R_m c_m$, so $w P_m c_m$. Likewise, $m P_w c_w$. But then c is not admissible. Individual rationality is immediate. □

Corollary 1. *We can write*

$$f(p) = \bigcup_{c \in \mathcal{C}} P_c(p)$$

That is, given q, the stable matchings are a finite union of closed polyhedra.

Proof. It suffices to show that for any stable $Q \in f(p)$, we have $Q \in P_c(p)$ for some c. This is immediate, because the profile of worst realized matches in $\text{supp}(Q)$ is an admissible cutoff profile. □

Corollary 2. *$f(p)$ is compact.*

Proof. Corollary 1 shows $f(p)$ is closed. Feasibility enforces $f(p)$ is bounded. By existence, $f(p)$ is nonempty. Thus $f(p)$ is compact. $\square$

4 Continuity

We now turn to the main results. The first result is that f is uhc everywhere and thus continuous when it is single-valued.

Theorem 1. *The stable matching correspondence $f : \mathcal{P} \rightrightarrows \mathcal{Q}$ is upper hemicontinuous everywhere. If $|f(p)| = 1$, then $f(\cdot)$ is continuous at p.*

Proof. Let $p^n \to p$ be a convergent sequence in $\mathcal{P}$. Let $Q^n \in f(p^n)$ be a convergent sequence such that $Q^n \to Q$. We wish to show $Q \in f(p)$. That is, we show Q is feasible, IR, and PS at p.

- **Feasibility.** Since row and column sums of Q are continuous functions of p, feasibility is preserved in the limit.
- **IR.** Suppose there is (m, w) such that $Q_{m,w} > 0$ but $\phi \succ_m w$. Then for large enough n, $Q^n_{m,w} > 0$, else $Q^n \not\to Q$, a contradiction to stability of Q^n. The w case is symmetric.
- **PS.** Suppose (m, w) is a blocking pair for Q. Then there exists (m', w') such that

  $$Q_{m,w'} > 0, \quad Q_{m',w} > 0, \quad w \succ_m w', \quad m \succ_w m'$$

  But then for large enough n, $Q^n_{m,w'} > 0$ and $Q^n_{m',w} > 0$, contradicting stability of Q^n.

When $|f(p)| = 1$ and $f(\cdot)$ is compact-valued, equivalence of uhc to continuity is standard. Let $p^n \to p$, and choose any $Q^n \in f(p^n)$. Since p^n is bounded, the corresponding matchings are bounded, and Q^n has a convergent subsequence. By uhc, its limit must lie in $f(p)$. Since $f(p) = \{Q\}$, every limit of a convergent subsequence is Q. Hence $Q^n \to Q$, as desired. $\square$

The second and more difficult result is that f is lhc at p if and only if $f(p)$ is single-valued.

Theorem 2. *The stable matching correspondence $f : \mathcal{P} \rightrightarrows \mathcal{Q}$ is lower hemicontinuous at p if and only if $|f(p)| = 1$.*

The proof is in the proceeding section.

4.1 Proof of Theorem 2

Supposing $f(p)$ is multi-valued, the following result identifies a cycle along which mass can flow while remaining stable.

First, we describe Baïou and Balinski (2002)'s man-improving algorithm. Starting from any stable matching, the algorithm either certifies man type m as optimal, or finds a directed cycle that can improve m. The usefulness of the algorithm (for our purpose) is that it generates a convex set of stable matches along a cycle; the welfare aspect and exact construction is unimportant for us. Readers may safely skip to Lemma 5.

Given a stable matching Q and mutually acceptable pair (m, w), call (m, w) man-stable if

$$\sum_{w' \succeq_m w} Q_{m,w'} = p_m$$

and woman-stable if

$$\sum_{m' \succeq_w m} Q_{m',w} = p_w$$

The algorithm constructs a chain of men as follows:

1. Let M^* be the subset of men whose allocations are already man-optimal. That is, $Q_m = Q_m^M$ for $m \in M^*$. Initialize M^* with all $m \in M$ such that $Q_{m,\phi} > 0$ (by 3). If $M^* = M$, stop.

   Otherwise, choose $m_0 \notin M^*$, and let w_0 be m_0 's worst realized partner in Q. Call (m_0, w_0) even; we will remove mass from even cells.

2. At an even cell (m_t, w_t), consider w_t's preference ranking. Consider men $m \notin M^*$ such that (m, w_t) is not man-stable. Choose the one most preferred by w_t, and call him m_{t+1}. (Existence is treated below.) Thus

   $$m_t \succ_{w_t} m_{t+1}$$

   and

   $$\sum_{w' \succeq_{m_{t+1}} w_t} Q_{m_{t+1},w'} < p_{m_{t+1}}$$

   The cell (m_{t+1}, w_t) is an odd cell; mass will be added to it.

   Suppose no such m_{t+1} exists. Then Baïou and Balinski (2002) (cf. pp. 678-679) show that all men encountered so far in the chain are man-optimal; add them to M^* and return to Step 1.

3. If m_{t+1} exists, let w_{t+1} be his worst realized partner in Q. Since (m_{t+1}, w_t) is not man-stable, m_{t+1} must have positive mass matched with someone strictly worse than w_t. Hence

   $$w_t \succ_{m_{t+1}} w_{t+1}, \quad Q_{m_{t+1},w_{t+1}} > 0.$$

   Call (m_{t+1}, w_{t+1}) the next even cell.

4. Continue alternating even and odd cells until a man or woman repeats. Then there is a simple

cycle

$$(m_1, w_1), (m_2, w_1), (m_2, w_2), \ldots (m_K, w_K), (m_1, w_K)$$

where moving mass along $(m_k, w_k) \to (m_{k+1}, w_k)$ improves the man side.

Formally, let

$$D = \sum_{k=1}^{K} (e_{m_{k+1}, w_k} - e_{m_k, w_k})$$

where $e_{m,w}$ is the basis vector in $\mathcal{Q}$ denoting a 1 in the (m, w) entry. Let $\bar{\alpha} = \min_{k \leq K} Q_{m_k, w_k}$, and set

$$Q^\alpha = Q + \alpha D, \quad 0 < \alpha < \bar{\alpha}.$$

$\{Q^\alpha\}$ is a set of man-side improving matches.

Baïou and Balinski (2002) also show that the constructed Q^α are stable. Briefly, for every involved man, mass moves from a less preferred woman to a more preferred woman. Thus any man-stable cell remains man-stable. For any involved woman w_k, mass moves from m_k to the less-preferred m_{k+1}. A cell (w_k, m) loses woman-stability only if m lies between m_k and m_{k+1}. But m_{k+1} was selected as the most-preferred man below m_k whose cell was not man-stable. Thus Q^α is stable.

Lemma 5. *Let $C = \{(m_1, w_1), (m_2, w_1), (m_2, w_2), \ldots (m_K, w_K), (m_1, w_K)\}$ be a cycle identified by Baïou and Balinski (2002)'s man-improving algorithm, and Q^α constructed as above. Then Q^α is stable.*

Proof. Omitted (Baïou and Balinski, 2002, cf. pp. 678-679.). □

The usefulness of the above lemma comes from the fact that Q^α are all stable and have the same support. Notice also that each pair in the cycle C has positive mass in Q^α.

Next, given the cutoff, we construct an undirected graph $G(c) = (M \cup W \cup \phi, E(c))$. That is, agents are nodes, and $E(c)$ are the edges. Consider the cycle C identified in Lemma 5, and let c be the binding cutoff profile for Q^α. Each pair $(m_1, w_1), (m_2, w_1), \ldots$ is an edge in $E(c)$, since there is positive mass on each; and this forms a cycle in $G(c)$.

Lemma 6. *Let $Q \in f(p)$, and let c be the cutoff profile of its worst realized matches. Then the connected component of the graph $G(c) = (M \cup W \cup \phi, E(c))$ containing ϕ is a tree.*

Proof. If c is admissible, then $(m, w) \in E(c)$ only if either $w = c_m$ or $m = c_w$. That is, edges are of the form (m, c_m) or (c_w, w).

Each vertex $i \in M \cup W$ contributes the edge (i, c_i) (re-ordering for M or W belonging). The vertex ϕ contributes no additional edges beyond these. A connected component with nodes $V \ni \phi$ thus contains at most $|V| - 1$ distinct edges. A connected component on $|V|$ vertices with $|V| - 1$ or fewer edges is a tree. □

We now turn to the proof of the main result. The intuition is that the C must be part of a connected component of $G(c)$ that does not contain ϕ. Thus this component has equal mass of both sides. If we introduce infinitesimal mass to the M side, the resulting stable matches cannot be in $P_c(p)$. Thus we find a discontinuous change in $f(p)$.

Proof of Theorem 2. Fix Q^α from Lemma 5, and let c be its binding cutoff profile. Denote the cycle identified by the man-improving algorithm as $C = \{(m_k, w_k), (m_{k+1}, w_k)\}_{k=1,\ldots,K}$. Each pair in the cycle has strictly positive mass, and they form a cycle in $G(c)$.

Let $p^n = p + \frac{1}{n} e_{m_1}$ (that is, adding shrinking mass to the first man in the cycle), so $p^n \downarrow p$. Toward a contradiction, suppose that f is lhc at p, so there exists a subsequence of p^n (abusing notation, denoted n) and sequence $Q^n \in f(p^n)$ with $Q^n \to Q^\alpha$. Let c^n be the cutoffs of the worst realized matches in Q^n.

For large enough n, we have $Q^n_{m,w} > 0$ whenever $(m, w) \in \text{supp}(Q^\alpha)$, in particular for $(m, w) \in C$. Thus each of these edges is in $E(c^n)$ for large enough n; recall that they form a cycle. Denote the component containing this cycle as $M^n \cup W^n$; since it contains a cycle, it does not include ϕ by Lemma 6. Thus

$$Q^n_{M^n,W^n} = \sum_{m \in M^n} p^n_m = \sum_{w \in W^n} p^n_w$$

as this component is fully matched.

Since $m_1 \in M^n$, we have

$$\left(\sum_{m \in M^n} p_m \right) + \frac{1}{n} = \sum_{w \in W^n} p_w$$

$$\implies \frac{1}{n} = \sum_{w \in W^n} p_w - \sum_{m \in M^n} p_m.$$

There are finitely many possible pairs (M^n, W^n), so $\sum_{w \in W^n} p_w - \sum_{m \in M^n} p_m$ has finitely many possible values. (Note p_w, p_m are not indexed by n.) But this contradicts that $\frac{1}{n} \to 0$. $\square$

5 Approximation of discrete markets by continuum markets

We note an application of continuity to the approximation of a large discrete market by a continuum market, in the style of Azevedo and Leshno (2016) and Che, Kim, and Kojima (2019). At a continuum market $(M, W, \succeq, p)$ and a particular sequence of discrete markets converging to it, every stable match in $f(p)$ is the limit of stable matches of the discrete markets if and only if $|f(p)| = 1$. Informally, it is safe to use the continuum market $(M, W, \succeq, p)$ to approximate large discrete markets if and only if the continuum market's stable match is unique.

We first define the notion of discrete marriage markets with shrinking agents. Define a sequence of finite (Gale-Shapley) marriage markets $(M, W, \succeq, k^n)$, where M and W are sets of discrete agent types. Preferences are strict between agent types, and indifferent among agents of the same type. The vector $k^n \in \mathbb{Z}^{M \cup W}$ is the counts of agents of each type.

Given $(M, W, \succeq, k^n)$ and p^n, we define a weakly stable matching μ analogously.

Definition 5. Given a discrete market $(M, W, \succeq, k^n)$, an aggregate matching is $\mu \in \mathbb{Z}_+^{(M \cup \phi) \times (W \cup \phi)}$, where $K_{m,w}$ represents the number of type m matched to type w. The aggregate matching K is **weakly stable** if:

- for all $m \in M$ and $w \in W$, if $K_{m,w} > 0$ then $K_{m,w} \succeq \phi$
- there does not exist a pair (m', w') such that

$$K_{m,w'} > 0, \quad K_{m',w} > 0$$

  and

$$w \succ_m w', \quad m \succ_w m'.$$

Let $a^n \to \infty$. For each finite economy, define the normalized population vector as

$$p_i^n = \frac{k_i^n}{a^n}.$$

Given a stable matching K^n for $(M, W, \succeq, k^n)$, define the aggregate matrix

$$\mu_{m,w}^n = \frac{1}{a^n} K_{m,w}^n$$

and analogously $\mu_{m,\phi}^n$ and $\mu_{\phi,w}^n$. A salient example is to let $k^n \to \infty$ and $a^n = ||k^n||_1$.

In this way, we can construct a sequence of discrete markets converging to a continuum market. When the limiting continuum market has a unique stable match, any sequence of stable matches in the discrete markets also converges in the expected way. We may also ask the opposite question: Given $f(p)$, can we approximate all $Q \in f(p)$ it by stable allocations of discrete markets? By applying our main result, the answer is yes if and only if $f(p) = \{Q\}$. In sum, the entire set of continuum stable matches $f(p)$ can be element-wise approximated by discrete markets if and only if $f(p)$ is a singleton.

It is immediate that $\mu^n \in f(p^n)$. That is, the aggregate matrix μ^n is a stable aggregate matching in the continuous model with a population vector p^n.

Lemma 7. *Let K^n be an aggregate stable matrix for a discrete market with population vector k^n. Fix a^n and normalized population p^n. Then $\mu^n \in f(p^n)$.*

Proof. Feasibility and IR are immediate. For PS, suppose (m, w) blocks K^n (in the continuum sense). Then there exists a pair (m', w') such that $K_{m,w'}^n > 0$ and $K_{m',w}^n > 0$, but $w \succ_m w'$ and $m \succ_w m'$. But then in the finite stable matching μ^n that yields K^n, the matchings (m, w') and (m', w) exist. Then (m, w) also blocks μ^n, a contradiction. $\square$

When a continuum market $(M, W, \succeq, p)$ has a unique stable matching Q, any sequence of stable matches of converging discrete markets converges to Q.

Proposition 2. *Fix a continuum market $(M, W, \succeq, p)$. Suppose $p^n := \frac{k_i^n}{a^n} \to p$, and K^n is a sequence of aggregate matrixes constructed from stable matches in respective finite economies given by $(M, W, \succeq, k^n)$. If $f(p) = \{Q\}$, then $\mu^n \to Q$.*

Proof. Each $K^n \in f(p^n)$. Since $p^n \to p$, the sequence K^n is bounded. By the Bolzano-Weierstrass theorem, consider a convergent subsequence (indexed m) $K^m \to \bar{K}$. Since $f(\cdot)$ is uhc, $\bar{K} \in f(p)$. But since $f(p)$ is a singleton, all convergent subsequences have the same limit Q. Hence the full sequence $K^n \to Q$. □

The following is this section's main result.

Proposition 3. *Fix $(M, W, \succeq, p)$. The following are equivalent:*

1. *$|f(p)| = 1$.*
2. *For every sequence of finite markets $(M, W, \succeq, k^n)$ and a^n such that $p^n := \frac{k_i^n}{a^n} \to p$, and for every $Q \in f(p)$, there exists a sequence of finite stable aggregate matrixes μ^n such that $\mu^n \to Q$.*

Proof. $1 \Rightarrow 2$ is a corollary of 2. We show the other direction. Toward a contradiction, suppose $f(p)$ is multivalued. By Theorem 2, $f(\cdot)$ is not lhc at p. Thus there exists some $Q \in f(p)$ and $\varepsilon > 0$, with a sequence $x^n \to p$ such that $f(x^n) \cap B_{2\varepsilon}(Q) = \emptyset$ for infinitely many n.

Since $f(\cdot)$ is uhc and compact-valued, we further have for each n, a neighborhood U^n of x^n such that

$$f(y) \cap B_\varepsilon(Q) = \emptyset \quad \forall y \in U^n.$$

Suppose not. Then there is a sequence $y^k \to x^n$ and $Y^k \in f(y^k)$ such that $\left\|Y^k - Q\right\| < \varepsilon$. Since Y^k is bounded, there is a convergent subsequence $Y^{k_\ell} \to \bar{Y}$. By uhc, $\bar{Y} \in f(x^n)$. But since $Y^{k_\ell}$is within ε of Q, this contradicts that $f(x^n) \cap B_{2\varepsilon}(Q) = \emptyset$.

Let $p^n \to p$ be such that

$$p^n = \frac{k^n}{a^n} \in U^n$$

We have $K^n \in f(p^n)$. However by construction,

$$f(p^n) \cap B_\varepsilon(Q) = \emptyset$$

and thus

$$\|K^n - Q\| \geq \varepsilon$$

as desired. □

Intuitively, uhc guarantees that limits of the discrete market stable matches are stable. However, generally there may be continuum stable aggregate matches that are not approximable by discrete markets. The following example also notes that side-optimal selection may not remove the problem.

Example 1. Let a continuum market be $N = \{m_1, m_2, w_1, w_2\}$ with preferences

m_1	m_2	w_1	w_2
w_1	w_2	m_2	m_1
w_2	w_1	m_1	m_2
ϕ	ϕ	ϕ	ϕ

and population $p = (1, 1, 1, 1)$. The stable matches are given by

$$Q^t = \begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix}$$

for $t \in [0, 1]$, with man-optimal stable match

$$Q^M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now consider $p^\varepsilon = (1 + \varepsilon, 1, 1, 1)$; that is, add ε mass to m_1. It is straight forward to see that the unique stable match in this market is

$$Q^\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with ε mass of m_1 unmatched.

Let a sequence of discrete markets be

$$N^n_{m_1} = n + 1, \quad N^n_i = n \quad \forall i \neq m_1$$

with $a_n = n$, so $p^n = (1 + \frac{1}{n}, 1, 1, 1) \to p$. Then

$$K^n = Q^{1/n} \to Q^W \neq Q^M$$

While setting a side-optimal matching as the target may not mitigate the issue, the sufficient condition for uniqueness in Baïou and Balinski (2002) suggests that there is likely a nearby market to $(M, W, \succeq, p)$ with a unique stable match. Indeed, p^ε itself fits the bill. Thus, if the application allows, approximation by a continuum market can be made robust by selecting a nearby market with a unique stable match.

6 Conclusion

We study a continuum stable marriage market. Building on the classic Gale and Shapley (1962) model, the additional topological structure allows us to derive continuity results of the correspondence from population vectors p to the set of stable matches $f(p)$. The correspondence is upper hemicontinuous everywhere, but lower hemicontinuous at p if and only if $f(p)$ is single-valued.

When p is composed of ones as in the classic discrete setting, $f(p)$ is generally multi-valued, offering formal explanation for the chaotic nature of the set of stable matchings. However, when p is drawn from real numbers, uniqueness and thus continuity is the typical case.

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